Real Analysis Problem Set 9, Part 1

04/28/2018

$\boxed{\text{Due 05/07, before class}}$

1. Let μ^* be an outer measure on X. Suppose $\mu^*(A) < +\infty$.
   Prove: $|\mu^*(A) - \mu^*(B)| \le \mu^*(A \triangle B)$, $\forall B \subset X$.

2. An outer measure μ^* on X is called regular if for any $A \subset X$, there is a Caratheodory measurable set $B \supset A$ s.t. $\mu^*(A) = \mu^*(B)$.
   Now suppose μ^* is a regular outer measure on X, and suppose $\mu^*(X) < +\infty$.
   Prove: A set $A \subset X$ is measurable $\Leftrightarrow \mu^*(A) + \mu^*(X \setminus A) = \mu^*(X)$.

3. Let $\mathcal{B}$ be a Boolean algebra on X, and $\mu_0 : \mathcal{B} \to [0, +\infty]$ a premeasure.
   Let μ^* and $\mathcal{F}_c$ be the outer measure and the σ-algebra that we constructed in the proof of Hahn-Kolmogorov thm (Page 5 of the notes).
   Now suppose ν is another measure on $\mathcal{F}_c$ s.t. $\nu|_{\mathcal{B}} = \mu_0$.
   Prove: (1) $\nu(A) \le \mu(A)$, $\forall A \in \mathcal{F}_c$
   (2) If $\mu(A) < +\infty$ and $A \in \mathcal{F}_c$, then $\mu(A) = \nu(A)$.
   (3) If μ_0 is σ-finite (i.e. $X = \bigcup_{n=1}^{\infty} X_n$, $\mu_0(X_n) < +\infty$), then $\mu = \nu$ on $\mathcal{F}_c$.
   [$\rightsquigarrow$ the uniqueness of product measure for two σ-finite measure spaces.]

$\boxed{\text{NOT required}}$ ↓

4. Let $\mathcal{B} = \{A_1 \cup \cdots \cup A_k : A_i = (a_i, b_i] \cap \mathbb{Q},\ k \in \mathbb{N}\}$.
   Prove: (1) $\mathcal{B}$ is a Boolean algebra.
   (2) The σ-algebra generated by $\mathcal{B}$ is $P(\mathbb{Q})$.
   (3). Define $\mu_0 : \mathcal{B} \to [0, +\infty]$ by $\mu_0(\phi) = 0$ and $\mu_0(A) = \infty$ for $A \ne \phi$.
   Then μ_0 is a premeasure, and there is more than one measure on $P(\mathbb{Q})$ whose restriction to $\mathcal{B}$ is μ_0.