05/03/2018

Real Analysis PSet 9, Part 2

Due 05/07, before class

1. Let (X,d) be a metric space, and $\mu: \mathcal{B}_X \to [0,+\infty]$ a Borel measure. Moreover, suppose $\mu(B(x,r)) < \infty$ for any $x \in X$, $r \in \mathbb{R}$. Let $A \subset X$ be any Borel set. Prove: $\forall \varepsilon > 0$, $\exists$ open set U, closed set F such that $F \subset A \subset U$, and $\mu(U \setminus F) < \varepsilon$.

   (Hint: Let $\mathcal{A} = \{A \subset X : A$ is a Borel set that satisfies the conclusion$\}$. Prove ① $\mathcal{A}$ contains all open sets. ② $\mathcal{A}$ is a σ-algebra.)

2. Let $F: \mathbb{R} \to \mathbb{R}$ be a monotone non-decreasing right-continuous function i.e. $f(x_0) = f(x_0+0)$, $\forall x_0 \in \mathbb{R}$. For any $A \subset \mathbb{R}$, define
$$\mu_F^*(A) = \inf\left\{\sum_{j=1}^{\infty}(F(b_j) - F(a_j)) : A \subset \bigcup_{j=1}^{\infty}(a_j, b_j]\right\}.$$
Prove: μ_F^* is a metric outer measure on $\mathbb{R}$, and $\mu_F^*((a,b]) = b-a$.

3. For any $A \subset \mathbb{R}$, define the outer measure $h_{1/2,\infty}$ by
$$h_{1/2,\infty}(A) := \inf\left\{\sum_{k=1}^{\infty}(b_k - a_k)^{1/2} : A \subset \bigcup_{k=1}^{\infty}[a_k, b_k]\right\}.$$
Prove: For any $0 < a < 1$, the interval $[0, a]$ is NOT measurable w.r.t. $h_{1/2,\infty}$.

4. Prove: (1) For any Borel set A in $\mathbb{R}^d$, and any $0 \le \alpha < \infty$,
$$\mathcal{H}^\alpha(A+x) = \mathcal{H}^\alpha(A), \quad \mathcal{H}^\alpha(\lambda A) = \lambda^\alpha \mathcal{H}^\alpha(A) \ (\lambda > 0)$$
(2) Any set in $\mathbb{R}^d$ is $\mathcal{H}^0$-measurable, and $\mathcal{H}^0 =$ the counting measure.

(3) For Borel set $A \subset \mathbb{R}$, what is $\mathcal{H}^1(A)$?