Course Information

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♣ **About the course:**

- **Course Name:** Semiclassical Microlocal Analysis
- **Instructor:** Zuoqin Wang
  - **Email:** wangzuoq@ustc.edu.cn
  - **Office:** 1601
- **Lecture Time:** Monday 15:55 pm - 17:30 pm
  Wednesday 19:30 pm - 21:05 pm
- **Lecture Room:** 5304
- **Webpage:**
  http://staff.ustc.edu.cn/~wangzuoq/Courses/20F-SMA/index.html
  [*You may check the following website for my old course notes:*
  http://staff.ustc.edu.cn/~wangzuoq/Courses/14F-Semiclassical/SMA.html *Note: The materials to be covered and the arrangement of topics in this semester will be slightly different from that one.*]
- **Homeworks:** Will be assigned every two weeks or so.
- **Office Hours:** By appointment (via email).
- **Your grades:** HWs + Essay

◊ **Notes and Reference books:**

- **Course Notes:**
  Will be uploaded to the Course Webpage after each lecture.
- **Reference Books:**
  The following are some nice reference books for this course.
  - *Semiclassical Analysis* by M. Zworkski
  - *Semi-Classical Analysis* by V. Guillemin and S. Sternberg
  - *Spectral Asymptotics in the Semi-Classical Limit* by M. Dimassi and J. Sjöstrand
  - *An Introduction to Semi-classical and Microlocal Analysis* by A. Martinez
  - *Harmonic Analysis in Phase Space* by G. Folland
  - *The Analysis of Linear Partial Differential Operators III* by L. Hörmander

LECTURE 1 — 09/21/2020
INTRODUCTION

1. What is and why

♡ **What is semiclassical microlocal analysis**

- **Q**: What is local?
  A: localize with respect to location in the space.
  - Configuration space: Euclidean space/manifolds.
  - Analysis: PDE
- **Q**: What is microlocal?
  A: Localize not only with respect to location in the space, but also with respect to cotangent directions at a given point.
  - Phase space: cotangent bundle
  - Analysis: Fourier/harmoic analysis
- **Q**: What is classical?
  A: Classical mechanics (in the framework of Hamilton).
  - Symplectic geometry: Hamilton flow, Lagrangian submanifolds ...
- **Q**: What is quantum?
  A: Quantum mechanics (in the framework of Schrödinger).
  - Spectral theory: eigenvalues, eigenfunctions, ...
- **Q**: What is semiclassical?
  A: A theory/method describing the intermediate between classical mechanics and quantum mechanics.

$\rightsquigarrow$ Roughly speaking, "semiclassical microlocal analysis is an investigation of the mathematical implications of the Bohr correspondence principle": a quantum system shall approximate its classical model at *formal* $\hbar \to 0$ (or high frequency = eigenvalue $\lambda \to \infty$) limit.

♠ **Why study?**

- Physics: provides the mathematical theory for
  - classical-quantum correspondence
  - particle-wave correspondence
  - geometric optics approximation
- Mathematics: develops methods to
  - apply harmonic analysis and symplectic geometry to PDEs,
  - study asymptotic behavior of eigenvalues/eigenfunctions
  - has broad applications to various subjects in mathematics including differential geometry, dynamical system, number theory etc

2. We will cover...

In this course we plan to cover

♣ **Backgrounds:**

- From physics: quantization and semiclassical limit
  - classical ⇝ phase space ⇝ symplectic
  - quantum ⇝ operators on Hilbert space ⇝ spectral theory
  - quantization: from classical description to quantum description
  - semiclassical limit: from quantum description to classical description
- From analysis: Fourier analysis, Method of stationary phase
  - Semiclassical Fourier transform
  $$\mathcal{F}_\hbar\varphi(\xi) := (\mathcal{F}\varphi)(\frac{\xi}{\hbar}) = \int_{\mathbb{R}^n} e^{-\frac{ix\cdot\xi}{\hbar}}\varphi(x)dx.$$
  - **The lemma of stationary phase:** The oscillatory integral
  $$I_\hbar = \int_{\mathbb{R}^n} e^{i\frac{\varphi(x)}{\hbar}}a(x)dx.$$
  has an asymptotic expansion as $\hbar \to 0$, with leading term
  $$I_\hbar = (2\pi\hbar)^{n/2}\sum_{d\varphi(p_i)=0} e^{i\frac{\varphi(p_i)}{\hbar}}e^{\frac{i\pi}{4}\mathrm{sgn}(d^2\varphi(p_i))}\frac{a(p_i)}{|\det d^2\varphi(p_i)|^{1/2}} + O(\hbar^{\frac{n}{2}+1}).$$
  This formula will play a crucial role in studying semiclassical limit.
- From geometry: Basic symplectic geometry
  - Symplectic manifolds:
    * Darboux: locally $T^*\mathbb{R}^n$
    * A. Weinstein: I like to think of symplectic geometry as playing the role in mathematics of a language which can facilitate communication between geometry and analysis
    * Hamiltonian mechanics
  - Lagrangian submanifolds
    * generating function: geometry ⇝ analysis
    * canonical relations, symplectic "category"
  - (we assume the knowledge of basic theory of manifolds)
  - (In some applications we assume the knowledge of basic Riemannian geometry)
    * The Laplace-Beltrami operator Δ as quantization of $\|\xi\|^2$
    * geodesic flow as Hamiltonian flow associated to the function $\|\xi\|$

◊ **Semiclassical pseudodifferential operators (as quantizations of symbols)**

- We can quantize the position function x_k and the momentum function ξ_k
$$x_k \rightsquigarrow Q_k = \text{multiplicaton by } x_k$$
$$\xi_k \rightsquigarrow P_k = \frac{\hbar}{\sqrt{-1}}\frac{\partial}{\partial x_k}$$
and thus can quantize the energy function to the Schrödinger operator
$$H = \frac{|\xi|^2}{2} + V(x) \rightsquigarrow \hat{H} = -\frac{\hbar^2}{2}\Delta + V(x).$$
But how to quantize more general functions like $x_1\xi_1$? Note:
$$x_1\xi_1 = \xi_1 x_1 \text{ but } Q_1 \circ P_1 \neq P_1 \circ Q_1!$$
Answer: use semiclassical Fourier transform
- Symbols and various semiclassical quantizations
  - For a function a which lies in some nice class (symbols),
$$a^L(x,\hbar D)(\varphi)(x) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} a(x,\xi)\varphi(y)dyd\xi,$$
$$a^R(x,\hbar D)(\varphi)(x) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} a(y,\xi)\varphi(y)dyd\xi,$$
$$a^W(x,\hbar D)(\varphi)(x) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} a(\frac{x+y}{2},\xi)\varphi(y)dyd\xi.$$
For example, if $a(x,\xi) = x_1\xi_1$, then
$$a^L = Q_1P_1, \quad a^R = P_1Q_1, \quad a^W = \frac{Q_1P_1 + P_1Q_1}{2}.$$
  - composition law: the composition of semiclassical PsDOs is still a semiclassical PsDO with computable symbols. In particular,
$$\left[a^W(x,\hbar D), b^W(x,\hbar D)\right] = \frac{\hbar}{i}\{a,b\}^W(x,\hbar D) + O(\hbar^3),$$
- Properties of PsDOs as operators
  - L^2-boundedness, compactness
  - Ellipticity, paramatrix (pseudo-inverse)
  - generalized Sobolev space
  - wave front set
- Global theory
  - Invariance under coordinate change
  - PsDOs on manifolds

♡ **Semi-classical Fourier integral operators (as quantizations of canonical relations)**

- First example: the propagator

$$U(t) = e^{-itQ/\hbar},$$

is a FIO that quantize the Hamilton flow ρ_t of $q(x,\xi)$:

Theorem 2.1 (Egorov's theorem)**.** *Let $b_t(x,\xi) = \rho_t^* a$ be the "classical flow-out" of the symbol $a(x,\xi)$ along the Hamiltonian flow ρ_t of $q(x,\xi)$. Then*

$$e^{itQ/\hbar} a^W(x,\hbar D) e^{-itQ/\hbar} = b_t^W(x,\hbar D) + O(\hbar).$$

- Oscillatory half densities associated with canonical relations
- semiclassical FIO, symbolic calculus

♠ **Applications to spectral theory**

In spectral theory we study eigenvalues/eigenfunction of operators that arise from analysis, geometry, physics etc. By definition they are non-trivial solutions to

$$Pu = \lambda u,$$

where λ is a real number called an eigenvalue of P, and u is called an eigenfunction associated to λ. In most interesting cases, the operator P is an elliptic (pseudo)differential operator, and the eigenvalues form an increasing discrete sequence

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

and the associated eigenfunctions can be chosen to form a normalized basis of the corresponding Hilbert space.

- The distribution of eigenvalues

  For example, consider the eigenvalue counting function

$$N(\lambda) = \#\{j \mid \lambda_j \leq \lambda\}.$$

We will prove

Theorem 2.2 (Weyl law)**.** *For the Schrodinger operator $\hat{H}$ (or for elliptic semi-classical pseudodifferential operators satisfying specific properties)*

$$N(\lambda) = \frac{1}{(2\pi\hbar)^n}\mathrm{Vol}(\{(x,\xi) \mid H(x,\xi) < \lambda\} + O(\hbar)).$$

- The spatial (in configuration space as well as in phase space) distribution of eigenfunctions

  For example, we will prove

Theorem 2.3 (Quantum ergodicity theorem)**.** *Suppose (M,g) is a Riemannian manifold whose geodesic flow is ergodic. Then there exists a density one sequence of L^2-normalized Laplacian eigenfunctions u_{j_k} such that for any f,*

$$\int_M |u_{j_k}|^2 f(x)dx \to ⨍_M f(x)dx$$