LECTURE 4 — 09/28/2020
THE FOURIER TRANSFORM

1. The Fourier transform on $\mathscr{S}, \mathscr{S}'$ and L^2

¶ Some notions.

We start with some abbreviations that will be used in this course.

- For $j = 1, \cdots, n$,
  - $\partial_j = \frac{\partial}{\partial x_j}$.
  - $D_j = \frac{1}{i}\partial_j$.
- For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$.
  - $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
  - $\alpha! = \alpha_1! \cdots \alpha_n!$.
  - $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.
  - $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.
  - $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$.

¶ Schwartz functions.

This is the "best" class of functions:

Definition 1.1. A function $\varphi \in C^\infty(\mathbb{R}^n)$ is called a *Schwartz function* (or a *rapidly decreasing function*) if

$$\sup_{\mathbb{R}^n} |x^\alpha \partial^\beta \varphi| < \infty \qquad (1)$$

for all multi-indices α and β.

Example. Any compactly supported smooth function on $\mathbb{R}^n$ is a Schwartz function.

Example. For any $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$, the function $\varphi(x) = e^{-\lambda|x|^2}$ is a Schwartz function.

Example. If φ is a Schwartz function, so are the functions $x^\alpha D^\beta \varphi$, $D^\alpha x^\beta \varphi$, where α, β are any multi-indices. In particular, the eigenfunctions of the Harmonic oscillator $\hat{H} = -\frac{\hbar^2}{2}\Delta + \frac{|x|^2}{2}$ that we get last time are all Schwartz functions.

We will denote the set of all Schwartz functions by $\mathscr{S}(\mathbb{R}^n)$, or by $\mathscr{S}$ for simplicity. Obviously $\mathscr{S}$ is an infinitely dimensional vector space, and

$$C_0^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n), \qquad \forall 1 \le p \le \infty.$$

To do analysis, we need to give a topology on the space $\mathscr{S}$. This can be done via semi-norms. Recall that a *semi-norm* on a vector space V is function $|\cdot| : V \to \mathbb{R}$ so that for all $\lambda \in C$ and all $u, v \in V$,

(1) (Absolute homogeneity) $|\lambda v| = |\lambda||v|$.
(2) (Triangle inequality) $|u+v| \leq |u| + |v|$.

On $\mathscr{S}$ one can define, for each pair of multi-indices α, β a semi-norm

$$|\varphi|_{\alpha,\beta} = \sup_{\mathbb{R}^n} |x^\alpha \partial^\beta \varphi|. \tag{2}$$

This is a separating family of semi-norms using which one can define a metric d on $\mathscr{S}$ via

$$d(\varphi, \psi) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|\varphi - \psi\|_k}{1 + \|\varphi - \psi\|_k}, \tag{3}$$

where for each $k \geq 0$,

$$\|\varphi\|_k = \max_{|\alpha|+|\beta| \leq k} \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \varphi|$$

is a norm on $\mathscr{S}$. The topology on $\mathscr{S}$ we are going to use is the metric topology induced by d. It is easy to see that $\varphi_j \to \varphi \quad$ in $\mathscr{S}$ if and only if

$$|\varphi_j - \varphi|_{\alpha,\beta} \to 0$$

as $j \to \infty$ for all α, β. With respect to this topology, $\mathscr{S}$ is a *Fréchet space.*

¶ The Fourier transform on $\mathscr{S}$.

Let $\varphi \in \mathscr{S}$ be a Schwartz function. By definition its Fourier transform $\mathcal{F}\varphi$ is

$$\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x) dx. \tag{4}$$

We list several basic properties of the Fourier transform whose proofs (which follows from direct computations) will be omitted.

Proposition 1.2. *For any $\varphi, \psi \in \mathscr{S}$ and any multi-index α, we have*

(1) $\mathcal{F}(x^\alpha \varphi) = (-1)^{|\alpha|} D_\xi^\alpha (\mathcal{F}(\varphi))$.
(2) $\mathcal{F}(D_x^\alpha \varphi) = \xi^\alpha \mathcal{F}(\varphi)$.
(3) $\int_{\mathbb{R}^n} \mathcal{F}\varphi(\xi)\psi(\xi) d\xi = \int_{\mathbb{R}^n} \varphi(x) \mathcal{F}\psi(x) dx$.
(4) For any $\lambda \in \mathbb{R}_+$, if we let $\varphi_\lambda(x) = \varphi(\lambda x)$, then $\varphi_\lambda \in \mathscr{S}$ and

$$\mathcal{F}\varphi_\lambda(\xi) = \frac{1}{\lambda^n} \mathcal{F}\varphi(\frac{\xi}{\lambda}).$$

(5) For any $a \in \mathbb{R}^n$, if we let $T_a\varphi(x) = \varphi(x+a)$, then $T_a\varphi \in \mathscr{S}$ and

$$(\mathcal{F}T_a\varphi)(\xi) = e^{ia\cdot\xi} \mathcal{F}\varphi(\xi).$$

Note that as a consequence of (1) and (2), we have

$$\varphi \in \mathscr{S} \Longrightarrow \mathcal{F}\varphi \in \mathscr{S}.$$

One can show that as a linear map, $\mathcal{F} : \mathscr{S} \to \mathscr{S}$ is continuous.

As another consequence, we can compute the Fourier transform of the Gaussian function. We have

$$\mathcal{F}(e^{-|x|^2/2}) = (2\pi)^{\frac{n}{2}} e^{-|\xi|^2/2}. \tag{5}$$

To see this we just notice

$$\mathcal{F}(e^{-|x|^2/2})(\xi) = \int_{\mathbb{R}^n} e^{-|x|^2/2 - ix\cdot\xi} dx.$$

Taking ξ derivative, we get

$$\begin{aligned} \frac{\partial}{\partial \xi_j}\mathcal{F}(e^{-\frac{1}{2}|x|^2})(\xi) &= -ix_j \int_{\mathbb{R}^n} e^{-\frac{1}{2}|x|^2 - ix\cdot\xi} dx = i\int_{\mathbb{R}^n} \frac{\partial}{\partial x_j}(e^{-\frac{1}{2}|x|^2}) e^{-ix\cdot\xi} dx \\ &= -i\int_{\mathbb{R}^n} e^{-\frac{1}{2}|x|^2} \frac{\partial}{\partial x_j}(e^{-ix\cdot\xi}) dx = -\xi_j \mathcal{F}(e^{-\frac{1}{2}|x|^2})(\xi). \end{aligned}$$

It follows

$$\mathcal{F}(e^{-|x|^2/2})(\xi) = Ce^{-|\xi|^2/2},$$

where

$$C = \mathcal{F}(e^{-|x|^2/2})(0) = \int_{\mathbb{R}} e^{-|x|^2/2} dx = (2\pi)^{n/2}.$$

Using these properties we can prove

Theorem 1.3. *$\mathcal{F} : \mathscr{S} \to \mathscr{S}$ is an isomorphism with inverse*

$$\mathcal{F}^{-1}\psi(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi) d\xi. \tag{6}$$

Proof. We have

$$\int \mathcal{F}\varphi(\xi)\psi(\lambda\xi) d\xi = \frac{1}{\lambda^n}\int \varphi(x)\mathcal{F}\psi(\frac{x}{\lambda}) dx = \int \varphi(\lambda x)\mathcal{F}\psi(x) dx.$$

Letting $\lambda \to 0$, we get

$$\psi(0)\int \mathcal{F}\varphi(\xi) d\xi = \varphi(0)\int \mathcal{F}\psi(x) dx.$$

In particular, if we take $\psi(\xi) = e^{-\frac{|\xi|^2}{2}}$, then we get

$$\varphi(0) = \frac{1}{(2\pi)^n}\int \mathcal{F}\varphi(\xi) d\xi. \tag{7}$$

Finally we replace φ by $T_a\varphi$ in the above formula, then

$$\varphi(a) = T_a\varphi(0) = \frac{1}{(2\pi)^n}\int e^{ia\cdot\xi}\mathcal{F}\varphi(\xi) d\xi.$$

This is exactly what we need. $\square$

As a consequence, we get the following Parseval's identity:

Corollary 1.4. *For* $\varphi \in \mathscr{S}$,

$$\|\varphi\|_{L^2}^2 = \frac{1}{(2\pi)^n} \|\mathcal{F}\varphi\|_{L^2}^2.$$

Proof. We have

$$\begin{aligned} \frac{1}{(2\pi)^n} \|\mathcal{F}\varphi\|_{L^2}^2 &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\varphi(\xi) \overline{\mathcal{F}\varphi(\xi)} d\xi \\ &= \int_{\mathbb{R}^n} \mathcal{F}\varphi(\xi) (\mathcal{F}^{-1}\overline{\varphi})(\xi) d\xi \\ &= \int_{\mathbb{R}^n} \varphi(x) (\mathcal{F}\mathcal{F}^{-1}\overline{\varphi})(x) dx \\ &= \|\varphi\|_{L^2}^2. \end{aligned}$$

□

¶ The Fourier transform on $\mathscr{S}'$.

Definition 1.5. Any linear continuous map $u : \mathscr{S} \to \mathbb{C}$ is called a *tempered distribution.* The space of tempered distributions is denoted by $\mathscr{S}'$.

So $\mathscr{S}'$ is the dual of $\mathscr{S}$. The topology of the space $\mathscr{S}'$ is defined to be th weak-* topology, so that $u_j \to u$ in $\mathscr{S}'$ if

$$u_j(\varphi) \to u(\varphi), \quad \forall \varphi \in \mathscr{S}.$$

Example. Here are some examples of tempered distributions:

- For any $a \in \mathbb{R}^n$, the *Dirac distribution* $\delta_a : \mathscr{S} \to \mathbb{C}$ defined by
$$\delta_a(\varphi) = \varphi(a)$$
is a tempered distribution.
- More generally, for any $a \in \mathbb{C}^n$ and any multi-index α, the map $u : \mathscr{S} \to \mathbb{C}$ defined by
$$u(\varphi) = D^\alpha \varphi(a)$$
is a tempered distribution.
- Any bounded continuous function ψ on $\mathbb{R}^n$ defines a tempered distribution by
$$u_\psi(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\psi(x) dx.$$
Note that if $\psi \in \mathscr{S}$, then the distribution defined by the above formula is non-zero unless $\psi = 0$. It follows that $\mathscr{S} \subset \mathscr{S}'$.

As we can see from the examples, a tempered distribution need not be a function. However, given a tempered distribution, we can still define some operations on them as if they are functions:

Definition 1.6. Let $u \in \mathscr{S}'$. We define

(1) $x^\alpha u \in \mathscr{S}'$ via $(x^\alpha u)(\varphi) = u(x^\alpha \varphi)$.
(2) $D^\alpha u \in \mathscr{S}'$ via $(D^\alpha u)(\varphi) = (-1)^{|\alpha|} u(D^\alpha \varphi)$.
(3) $\mathcal{F}u \in \mathscr{S}'$ via $(\mathcal{F}u)(\varphi) = u(\mathcal{F}\varphi)$.

Remark. One can check that if ψ is a bounded smooth function, then the distributions $x^\alpha u_\psi$, $D^\alpha u_\psi$, $\mathcal{F}(u_\psi)$ (here we assume ψ is a Schwartz function) coincides with $u_{x^\alpha \psi}$, $u_{D^\alpha \psi}$ and $u_{\mathcal{F}\psi}$. In other words, these operations on tempered distributions, when restricted to usual functions, are the same operators that we are familiar with. So in these cases, for simplicity we will simply denote $x^\alpha u_\psi$, $D^\alpha u_\psi$, $\mathcal{F}(u_\psi)$ by $x^\alpha \psi$, $D^\alpha \psi$ and $\mathcal{F}(\psi)$.

¶ The Fourier transform on $L^2(\mathbb{R}^n)$.

We can also view any L^2 function ψ as a tempered distribution by using the same formula above, namely,

$$u_\psi(\varphi) := \int_{\mathbb{R}^n} \varphi(x)\psi(x)dx.$$

By this way we gets a natural embedding

$$C_c^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n),$$

where one can check that all the inclusion maps are continuous. In particular, for any $\psi \in L^2$, one can define its Fourier transform $\mathcal{F}\psi$, in the sense of distribution. It turns out that

Proposition 1.7 (Plancherel's Theorem)**.** *If $\psi \in L^2(\mathbb{R}^n)$, then the tempered distribution $\mathcal{F}\psi$ is also in $L^2(\mathbb{R}^n)$, and we still have the Parsevel's identity*

$$\|\psi\|_{L^2}^2 = \frac{1}{(2\pi)^n}\|\mathcal{F}\psi\|_{L^2}^2.$$

Proof. By Cauchy-Schwartz inequality, for any $\varphi \in \mathscr{S}$ we have

$$|\mathcal{F}(\psi)(\varphi)| = |\psi(\mathcal{F}\varphi)| \le \|\psi\|_{L^2} \cdot \|\mathcal{F}\varphi\|_{L^2} = (2\pi)^{n/2}\|\psi\|_{L^2} \cdot \|\varphi\|_{L^2}.$$

Since $\mathscr{S}$ is dense in $L^2(\mathbb{R}^n)$, the same inequality holds for $\varphi \in L^2(\mathbb{R}^n)$. In other words, $\mathcal{F}(\psi)$ is a bounded linear map on $L^2(\mathbb{R}^n)$. By Riesz representation theorem, $F\psi$ is represented by an L^2 function.

Moreover, the same formula above also implies that as an L^2-function,

$$\|\mathcal{F}\psi\|_{L^2} \le (2\pi)^{n/2}\|\psi\|_{L^2}.$$

It follows that

$$\|\mathcal{F}\mathcal{F}\psi\|_{L^2} \le (2\pi)^n\|\psi\|_{L^2}.$$

But a direct computation shows $\mathcal{F}\mathcal{F}\psi(x) = (2\pi)^n\psi(-x)$ for Schwartz functions, and thus for L^2 functions by the same arguments above. So $\|\mathcal{F}\mathcal{F}\psi\|_{L^2} = (2\pi)^n\|\psi\|_{L^2}$. The Parsevel's identity follows. □

2. Several Gaussian integrals

For the future purpose, we need to compute the Fourier transform of the Gaussian function $e^{-\frac{1}{2}x^TQx}$. For simplicity we do the computation using the standard Fourier transform. One can easily rewrite these results in the semiclassical setting.

Theorem 2.1. *Let Q be any real, symmetric, positive definite $n \times n$ matrix, then*

$$\mathcal{F}(e^{-\frac{1}{2}x^TQx})(\xi) = \frac{(2\pi)^{n/2}}{(\det Q)^{1/2}} e^{-\frac{1}{2}\xi^TQ^{-1}\xi}. \tag{8}$$

Proof. Let's first assume $n = 1$, so that $Q = q$ is a positive number. By a change of variable argument from (5) [or by repeating the proof if you want], one easily gets

$$\mathcal{F}(e^{-\frac{1}{2}qx^2})(\xi) = \frac{(2\pi)^{1/2}}{q^{1/2}} e^{-\frac{1}{2}\xi^2/q}.$$

Next suppose Q is a diagonal matrix. Then the left hand side of (8) is a product of n one-dimensional integrals, and the result follows from the one-dimensional case.

For the general case, one only need to choose an orthogonal matrix O so that

$$O^TQO = D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n),$$

where λ_i's are the eigenvalues of Q. The results follows from change variables from x to $y = O^{-1}x$:

$$\begin{aligned}\mathcal{F}(e^{-\frac{1}{2}x^TQx})(\xi) &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}x^TQx - ix\cdot\xi} dx = \int_{\mathbb{R}^n} e^{-\frac{1}{2}y^TDy - iy\cdot O^T\xi} dy \\ &= \frac{(2\pi)^{n/2}}{(\det D)^{1/2}} e^{-\frac{1}{2}\xi^TOD^{-1}O^T\xi} = \frac{(2\pi)^{n/2}}{(\det Q)^{1/2}} e^{-\frac{1}{2}\xi^TQ^{-1}\xi}.\end{aligned}$$

□

Now suppose Q is an $n \times n$ complex matrix such that $Q = Q^T$ and $\mathrm{Re}(Q)$ is positive definite. We observe that the integral on the left hand side of (8) is an analytic function of the entries $Q_{ij} = Q_{ji}$ of Q in the region $\mathrm{Re}Q > 0$. The same is true for the right hand side of (8), as long as we choose $\det^{1/2} Q$ to be the branch such that $\det^{1/2} Q > 0$ for real positive definite Q. So we immediately get

Theorem 2.2. *Let Q be any $n \times n$ complex matrix such that $Q = Q^T$ and $\mathrm{Re}(Q)$ is positive definite, then*

$$\mathcal{F}(e^{-\frac{1}{2}x^TQx})(\xi) = \frac{(2\pi)^{n/2}}{(\det Q)^{1/2}} e^{-\frac{1}{2}\xi^TQ^{-1}\xi}. \tag{9}$$

Remark. Another way to describe this choice of $\det^{1/2}$: Since $Q = Q^T$, for any complex vector $w \in \mathbb{C}^n$ we have

$$\mathrm{Re}(\bar{w}^TQw) = \bar{w}^T(\mathrm{Re}Q)w.$$

It follows that all eigenvalues $\lambda_1, \cdots, \lambda_n$ of Q has positive real part. Then

$$(\det Q)^{1/2} = \lambda_1^{1/2} \cdots \lambda_n^{1/2},$$

where $\lambda_j^{1/2}$ is the square root of λ_j that has positive real part.

Finally let Q be any real, symmetric, non-singular $n \times n$ matrix. Next time we will need the Fourier transform of the function $e^{\frac{i}{2}x^T Qx}$. Note that the function $e^{\frac{i}{2}x^T Qx}$ is not a Schwartz function, so we are really thinking of it as a tempered distribution and calculate its Fourier transform.

Recall that the *signature* of the matrix Q is

$$\text{sgn}(Q) = N^+(Q) - N^-(Q), \tag{10}$$

where $N^{\pm}(Q) =$ the number of positive/negative eigenvalues of Q.

Theorem 2.3. *For any real, symmetric, non-singular $n \times n$ matrix Q,*

$$\mathcal{F}(e^{\frac{i}{2}x^T Qx}) = \frac{(2\pi)^{n/2} e^{i\frac{\pi}{4}\text{sgn}(Q)}}{|\det Q|^{\frac{1}{2}}} e^{-\frac{i}{2}\xi^T Q^{-1}\xi}. \tag{11}$$

Proof. Let's first assume $n = 1$, so that $Q = q \neq 0$ is a non-zero real number. For any $\varepsilon > 0$ we let $q_\varepsilon = q + i\varepsilon$. Applying Theorem 2.2 we get

$$\mathcal{F}(e^{\frac{i}{2}q_\varepsilon x^2}) = \mathcal{F}(e^{-\frac{1}{2}(\varepsilon - iq)x^2}) = \frac{(2\pi)^{1/2}}{(\varepsilon - iq)^{1/2}} e^{-\frac{1}{2}(\varepsilon - iq)\xi^2},$$

the square root is described after Theorem 2.2. Note that when $\varepsilon \to 0+$, we have

$$(\varepsilon - iq)^{1/2} \longrightarrow \left\{ \begin{array}{ll} \sqrt{q}e^{-\frac{i\pi}{4}}, & q > 0 \\ \sqrt{-q}e^{\frac{i\pi}{4}}, & q < 0 \end{array} \right\} = \sqrt{|q|}e^{-\frac{i\pi}{4}\text{sgn}(q)},$$

so the conclusion follows.

For $n > 1$, one just proceed as before: first handle the case of diagonal matrices, then convert the general case to the diagonal case via the diagonalization trick. □