LECTURE 5 — 09/30/2020
THE METHOD OF STATIONARY PHASE

1. The method of stationary phase: simple models

¶ Asymptotic series.

When talking about the asymptotic behavior as $\hbar \to 0$[1], we will use the notations "big O" and "small o" as in mathematical analysis, namely for functions f, g which are depending on $\hbar$,

- $f = O(g)$ means "$\exists$ constant $C > 0$ such that $|f| \le C|g|$ for all small $\hbar$".
- $f = o(g)$ means "as $\hbar \to 0$, the quotient $f/g \to 0$".

We will use the following conceptions/notations from asymptotic analysis:

Definition 1.1. Let $f = f(\hbar)$.

(1) We say $f \sim \sum_{k=0}^{\infty} a_k \hbar^k$ if for each non-negative integer N,

$$f - \sum_{k=0}^{N} a_k \hbar^k = O(\hbar^{N+1}), \qquad \hbar \to 0.$$

(2) We say $f = O(\hbar^\infty)$ if $f \sim 0$, i.e. $f = O(\hbar^N)$ for all N.

Remark. When we write $f \sim \sum_{k=0}^{\infty} a_k \hbar^k$, we don't require the series $\sum_{k=0}^{\infty} a_k \hbar^k$ to converge! Moreover, even if the series converges at a point, it need not converge to the value of f at that point. For example, for any smooth function $f = f(\hbar)$, its Taylor series is an asymptotic series,

$$f(\hbar) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \hbar^k.$$

However, the series converges for all small $\hbar$ to the value $f(\hbar)$ only if f is analytic at 0.

Example. Here is a more concrete example: Consider the function $f(\hbar) = \begin{cases} e^{-1/\hbar}, & \hbar > 0 \\ 0, & \hbar \le 0. \end{cases}$ which is widely used in building cut-off functions. Then f is smooth everywhere, but it is not analytic at 0. Moreover, by induction one can easily prove $f^k(0) = 0$ for all k. It follows that $f \sim 0$, i.e. $f = O(\hbar^\infty)$.

[1] In this course, when we say "as $\hbar \to 0$", we always means "as $\hbar \to 0+$".

We can perform standard operations on asymptotic series. For example, if

$$f(\hbar) \sim \sum a_j \hbar^j, \qquad g(\hbar) \sim \sum b_j \hbar^j,$$

then we will have

$$f(\hbar) \pm g(\hbar) \sim \sum (a_j \pm b_j)\hbar^j \qquad \text{and} \qquad f(\hbar)g(\hbar) \sim \sum c_j \hbar^j,$$

where $c_j = \sum_{l=0}^{j} a_l b_{j-l}$. Similarly one can calculate the quotient of two asymptotic series: If $b_0 \neq 0$, then

$$f(\hbar)/g(\hbar) \sim \sum d_j \hbar^j,$$

where d_j's are defined iteratively via $d_0 = a_0/b_0$ and $d_j = b_0^{-1}(a_j - \sum_{l=0}^{j-1} d_l b_{j-l})$.

¶ Oscillatory integrals.

Very often in semiclassical analysis we will need to evaluate the asymptotic behavior of the *oscillatory integrals* of the form

$$I_\hbar = \int_{\mathbb{R}^n} e^{i\frac{\varphi(x)}{\hbar}} a(x)dx, \tag{1}$$

where $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ is called the *phase*, and $a \in C_c^\infty(\mathbb{R}^n, \mathbb{C})$ is called the *amplitude*. The method of stationary phase is the correct tool for this purpose.[2]

To illustrate, let's start with two extremal cases:

- Suppose $\varphi(x) = c$ is a constant. Then
$$I_\hbar = e^{ic/\hbar}A$$
which is fast oscillating as $\hbar \to 0$, where $A = \int_{\mathbb{R}^n} a(x)dx$ is a constant independent of $\hbar$.
- Suppose $n = 1$ and $\varphi(x) = x$. Then by definition $I_\hbar = \mathcal{F}(a)(-\frac{1}{\hbar})$. Since a is compactly supported, $\mathcal{F}(a)$ is Schwartz. It follows $I_\hbar = O(\hbar^N)$ for any N, i.e.
$$I_\hbar = O(\hbar^\infty).$$

Intuitively, in the second case the exponential $\exp(i\frac{x}{\hbar})$ oscillates fast in any interval of x for small $\hbar$, so that many cancellations take place and thus we get a function rapidly decreasing in $\hbar$. This is in fact the case at any point x which is not a critical point of φ. On the other hand, near a critical point of x the phase function φ doesn't change much, i.e. "looks like" a constant, so that we are in case 1. According to these intuitive observation, we expect that the main contributions to $I_\hbar$ should arise from the critical points of the phase function φ.

[2]Here we assume the phase function φ is real-valued. In the case the phase function is complex, there is a similar way to evaluate the asymptotic of the integral: the *method of steepest descent*.

¶ Non-stationary phase.

Proposition 1.2. *Suppose the phase function φ has no critical point in a neighborhood of the support of a. Then*

$$I_\hbar = O(\hbar^\infty). \tag{2}$$

Proof. Let χ be a smooth cut-off function such that

- χ is identically one on the support of a
- φ has no critical point on the support of χ.

Then $a(x) = \chi(x)a(x)$, and $\frac{\chi(x)}{|\nabla\varphi(x)|^2}$ is a smooth function. The trick (which is standard in semiclassical microlocal analysis) is to introduce an operator L given by

$$Lf(x) = \sum_j \frac{\chi(x)\partial_j\varphi(x)}{|\nabla\varphi(x)|^2}\partial_j f(x).$$

Then it is easy to check

$$L(e^{i\frac{\varphi(x)}{\hbar}})(x) = \frac{\chi}{|\nabla\varphi|^2}\sum_j \partial_j\varphi\frac{i\partial_j\varphi}{\hbar}e^{i\frac{\varphi(x)}{\hbar}} = \frac{i}{\hbar}\chi e^{i\frac{\varphi(x)}{\hbar}}.$$

It follows

$$I_\hbar = \frac{\hbar}{i}\int_{\mathbb{R}^n} L(e^{i\frac{\varphi(x)}{\hbar}})a(x)dx = \frac{\hbar}{i}\int_{\mathbb{R}^n} e^{i\frac{\varphi(x)}{\hbar}}(L^*a)(x)dx,$$

where L^* is the adjoint of L, explicitly given by

$$L^*f(x) = -\sum_j \partial_j\left(\frac{\chi(x)\partial_j\varphi}{|\nabla\varphi(x)|^2}f(x)\right).$$

Repeating this N times, we get

$$I_\hbar = \left(\frac{\hbar}{i}\right)^N \int_{\mathbb{R}^n} e^{i\frac{\varphi(x)}{\hbar}}(L^*)^N a(x)dx = O(\hbar^N).$$

□

¶ The stationary phase formula for quadratic phase.

Now we consider phase functions φ with *very simple* critical points, in which case one can imagine that the main contribution to the oscillatory integral comes from the critical point. We start with a model, i.e. $\varphi(x) = \frac{1}{2}x^TQx$ for some non-singular real symmetric $n \times n$ matrix Q.

Theorem 1.3 (Stationary phase for non-singular quadratic phase)**.** *Let Q be a real, symmetric, non-singular $n \times n$ matrix. For any $a \in C_c^\infty(\mathbb{R}^n)$, one has*

$$\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}x^TQx}a(x)dx \sim (2\pi\hbar)^{n/2}\frac{e^{i\frac{\pi}{4}\mathrm{sgn}(Q)}}{|\det Q|^{1/2}}\sum_k \hbar^k\frac{1}{k!}\left(-\frac{i}{2}p_{Q^{-1}}(D)\right)^k a(0), \tag{3}$$

where for a matrix $A = (a_{ij})$, we denote $p_A(D) = \sum a_{kl}D_kD_l$.

Proof. Recall Theorem 2.3 in Lecture 4: If $\varphi(x) = e^{\frac{i}{2}x^TQx}$, then

$$\mathcal{F}\varphi(\xi) = \frac{(2\pi)^{n/2}e^{i\frac{\pi}{4}\mathrm{sgn}(Q)}}{|\det(Q)|^{1/2}}e^{-\frac{i}{2}\xi^TQ^{-1}\xi}.$$

Now we apply the multiplication formula

$$\int \mathcal{F}\varphi(\xi)\psi(\xi)d\xi = \int \varphi(x)\mathcal{F}\psi(x)dx$$

to get

$$\int_{\mathbb{R}^n} \mathcal{F}a(\xi)e^{\frac{i}{2}\xi^T(-\hbar Q^{-1})\xi}d\xi = \frac{(2\pi)^{n/2}e^{i\frac{\pi}{4}\mathrm{sgn}(-\hbar Q^{-1})}}{|\det(-\hbar Q^{-1})|^{1/2}}\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}x^TQx}a(x)dx,$$

i.e.

$$\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}x^TQx}a(x)dx = \frac{\hbar^{n/2}}{(2\pi)^{n/2}}\frac{e^{i\frac{\pi}{4}\mathrm{sgn}(Q)}}{|\det Q|^{1/2}}\int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\xi^TQ^{-1}\xi}\mathcal{F}a(\xi)d\xi.$$

Using the Taylor's expansion formula for the exponential function, we see that for any non-negative integer N, the difference

$$\int_{\mathbb{R}^n} e^{-\frac{i\hbar}{2}\xi^TQ^{-1}\xi}\mathcal{F}a(\xi)d\xi - \sum_{k=0}^{N}\frac{1}{k!}(-\frac{i\hbar}{2})^k\int_{\mathbb{R}^n}(\xi^TQ^{-1}\xi)^k\mathcal{F}a(\xi)d\xi$$

is bounded by (a multiple of)

$$\hbar^{N+1}\frac{1}{2^{N+1}(N+1)!}\int_{\mathbb{R}^n}|(\xi^TQ^{-1}\xi)^k\mathcal{F}a(\xi)|d\xi.$$

So the conclusion follows from

Lemma 1.4. *For any $a \in \mathscr{S}$ and any polynomial p,*

$$\mathcal{F}(p(D)a)(\xi) = p(\xi)\mathcal{F}a(\xi).$$

Proof. This is just a consequence of the fact $\mathcal{F}(D^\alpha a) = \xi^\alpha\mathcal{F}a$ and the linearity of $\mathcal{F}$. □

which implies

$$(\xi^TQ^{-1}\xi)^k\mathcal{F}a(\xi) = \mathcal{F}(p_{Q^{-1}}(D)^ka)(\xi),$$

together with the Fourier inversion formula, which implies

$$p_{Q^{-1}}(D)^ka(0) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(\xi^TQ^{-1}\xi)^k\mathcal{F}a(\xi)d\xi.$$

This completes the proof. □

Remark. By calculating more carefully, one can prove that for any N, the remainder

$$R_N := \int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}x^TQx}a(x)dx - (2\pi\hbar)^{n/2}\frac{e^{i\frac{\pi}{4}\mathrm{sgn}(Q)}}{|\det Q|^{1/2}}\sum_{k=0}^{N-1}\hbar^k\frac{1}{k!}\left(-\frac{i}{2}p_{Q^{-1}}(D)\right)^k a(0)$$

is controlled by an explicit upper bound

$$R_N \leq C_N \sup_{|\alpha|\leq 2N+n+1}|\partial^\alpha a|.$$

2. The method of stationary phase: general case

¶ Morse lemma.

We first introduce some standard conceptions in global analysis:

Definition 2.1. Let φ be a smooth function.

(1) A point p is called a *critical point* of φ if
$$\nabla\varphi(p) = 0.$$

(2) A critical point p of φ is called *non-degenerate* if the Hessian matrix $d^2\varphi(p)$ is non-degenerate, i.e.
$$\det d^2\varphi(p) = \det\left[\frac{\partial^2\varphi}{\partial x_i \partial x_j}(p)\right] \neq 0.$$

(3) A smooth function is called a *Morse function* if all of its critical points are non-degenerate.[3]

To study the stationary phase expansion for more general phase functions, one first convert the general (non-degenerate) phase function to a quadratic one by using the Morse lemma:

Theorem 2.2 (Morse Lemma, version 1). *Let $\varphi \in C^\infty(\mathbb{R}^n)$. Suppose p is a non-degenerate critical point of φ. Then there exists a neighborhood U of 0, a neighborhood V of p and a diffeomorphism $\rho : U \to V$ so that $\rho(0) = p$ and*
$$\rho^*\varphi(x) = \varphi(p) + \frac{1}{2}(x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_n^2), \tag{4}$$
where r is the number of positive eigenvalues of the Hessian matrix $d^2\varphi(p)$.

¶ The stationary phase formula for general phase.

As we have seen, only the critical points of the phase function φ give an essential contribution to the oscillatory integral
$$I_\hbar = \int_{\mathbb{R}^n} e^{i\frac{\varphi(x)}{\hbar}} a(x)dx.$$
In what follows we will assume that $\varphi \in C^\infty(\mathbb{R}^n)$ admits only non-degenerate critical points in a neighborhood of the support of a. Since non-degenerate critical points must be discrete, φ has only finitely many critical points in the support of a. Thus one can find a partition of unity $\{U_i \mid 1 \le i \le N+1\}$ of the support of a so that each U_i, $1 \le i \le N$, contains exactly one critical point p_i of φ, and U_{N+1} contains

[3]Note that non-degenerate critical points must be discrete. In more subtle examples where the critical points are not necessary discrete, but still nice enough, say, form smooth manifolds, one has an analogous conception, namely the *Morse-Bott function*. Many results for Morse functions can be generalized to Morse-Bott functions.

no critical points of φ. A partition of unity arguments converts the asymptotic behavior of $I_\hbar$ to the sum of the asymptotic behavior of

$$I_\hbar^{p_i} = \int_{U_i} e^{i\frac{\varphi(x)}{\hbar}} \chi_i(x) a(x) dx,$$

where χ_i is a function compactly supported in U_i and equals 1 identically near p_i. According to the Morse lemma, for each U_i (one can always shrink U_i in the above integral if necessary) one can find a diffeomorphism $\rho : V_i \to U_i$ so that $\rho(0) = p_i$ and

$$\rho^*\varphi(x) = \varphi(p_i) + \frac{1}{2}(x_1^2 + \cdots + x_{r_i}^2 - x_{r_i+1}^2 - \cdots - x_n^2) =: \psi_{p_i}(x),$$

where r_i is the number of positive eigenvalues of the matrix $d^2\varphi(p_i)$. It follows from the change of variable formula that

$$I_\hbar^{p_i} = \int_{V_i} e^{i\frac{\psi_{p_i}(x)}{\hbar}} a(\rho(x))\chi_i(\rho(x))|\det d\rho(x)| dx.$$

Moreover, since $\psi_{p_i}(x)$ has a unique non-degenerate critical point at 0, modulo $O(\hbar^\infty)$ one has

$$I_\hbar^{p_i} = \int_{\mathbb{R}^n} e^{i\frac{\psi_{p_i}(x)}{\hbar}} a(\rho(x))|\det d\rho(x)| dx + O(\hbar^\infty). \tag{5}$$

The asymptotic of this integral is basically given by Theorem 1.3.

In particular, to get the leading term in the asymptotic expansion above, one only need to evaluate $|\det d\rho(0)|$, which follows from

Lemma 2.3. *Let ψ, φ be smooth functions defined on V and U respectively, and suppose 0 is a non-degenerate critical point of ψ, and p a non-degenerate critical point of φ. If $\rho : V \to U$ a diffeomorphism so that $\rho(0) = p$ and $\rho^*\varphi = \psi$. Then*

$$d\rho^T(0) d^2\varphi(p) d\rho(0) = d^2\psi(0). \tag{6}$$

Proof. We start with the equation $\varphi \circ \rho = \psi$. Taking derivatives of both sides one gets

$$\frac{\partial \varphi}{\partial y_k}(\rho(x))\frac{\partial \rho_k}{\partial x_j} = \frac{\partial \psi}{\partial x_j}.$$

Taking the second order derivatives at 0 of both sides, and using the conditions $\rho(0) = p$ and $\nabla\varphi(p) = 0$, one gets

$$\frac{\partial^2 \varphi}{\partial y_k \partial y_l}(p)\frac{\partial \rho_l}{\partial x_i}(0)\frac{\partial \rho_k}{\partial x_j}(0) = \frac{\partial^2 \psi}{\partial x_i \partial x_j}(0),$$

in other words

$$d\rho^T(0) d^2\varphi(p) d\rho(0) = d^2\psi(0).$$

□

It follows that in our case,

$$|\det d\rho(0)| = |\det d^2\varphi(p_i)|^{-1/2}. \tag{7}$$

As a result, we conclude

Theorem 2.4. *As $\hbar \to 0$, the oscillatory integral has the asymptotic expansion*

$$I_\hbar \sim \sum_{d\varphi(p_i)=0} I_\hbar^{p_i}, \tag{8}$$

where

$$I_\hbar^{p_i} \sim (2\pi\hbar)^{n/2} e^{i\frac{\varphi(p_i)}{\hbar}} e^{\frac{i\pi}{4}\mathrm{sgn}(d^2\varphi(p_i))} \frac{a(p_i)}{|\det d^2\varphi(p_i)|^{1/2}} \sum_{j\geq 0} \hbar^j L_j(a)(p_i), \tag{9}$$

where $L_j = L_j(x, D)$ is a differential operator (of order $2j$ which depends on the phase function φ) with $L_0 = 1$. In particular, the leading term is

$$I_\hbar = (2\pi\hbar)^{n/2} \sum_{d\varphi(p_i)=0} e^{i\frac{\varphi(p_i)}{\hbar}} e^{\frac{i\pi}{4}\mathrm{sgn}(d^2\varphi(p_i))} \frac{a(p_i)}{|\det d^2\varphi(p_i)|^{1/2}} + O(\hbar^{\frac{n}{2}+1}). \tag{10}$$

Remark. We have [c.f. L. Hörmander, *The Analysis of Linear Partial Differential Operators* Vol. 1, section 7.7.]

- One can write down an explicit formula for all these L_j's.
- There are more complicated version of the stationary phase expansion where the phase function is allowed to have degenerate critical points.

Appendix: A proof of Morse Lemma

We shall prove the following form of the Morse lemma which is obviously equivalent to the version stated above:

Theorem 2.5 (Morse Lemma, version 2)**.** *Let φ_0 and φ_1 be smooth functions on $\mathbb{R}^n$ such that*

- $\varphi_0(0) = \varphi_1(0) = 0$,
- $\nabla\varphi_0(0) = \nabla\varphi_1(0) = 0$.
- $d^2\varphi_0(0) = d^2\varphi_1(0)$ *is non-degenerate.*

Then there exist neighborhoods U_0 and U_1 of 0 and a diffeomorphism $\rho : U_0 \to U_1$ such that $\rho(0) = 0$ and $\rho^*\varphi_1 = \varphi_0$.

Proof. There are many different proofs of Morse Lemma. Here we provide a proof via the so-called *Moser's trick*, which is not very commonly appeared in literature. The idea is the following: To construct a diffeomorphism $\rho : U_0 \to U_1$ such that $f^*\varphi_1 = \varphi_0$, we will construct smooth family of functions φ_t connecting φ_0 and φ_1, and construct a time-dependent vector field Ξ_t so that the flow ρ_t satisfies the stronger relation

$$\rho_t^* \varphi_t = \varphi_0. \tag{11}$$

Of course if this is done, then the time-1 flow map $\rho = \rho_1$ is what we are looking for. Note that the equation (11) is equivalent to

$$0 = \frac{d}{dt}\rho_t^* \varphi_t = \rho_t^*(\dot{\varphi}_t + \mathcal{L}_{\Xi_t}\varphi_t).$$

So to construct such a vector field Ξ_t, it is enough to solve the equation $\mathcal{L}_{\Xi_t}\varphi_t = -\dot{\varphi}_t$. Note that to guarantee the existence of time-1 flow (at least locally near 0), we will require $\Xi_t(0) = 0$.

We apply Moser's trick as described above. To do so we let $\varphi_t = (1-t)\varphi_0 + t\varphi_1$. Then it is enough to find a time-dependent vector field Ξ_t such that $\Xi_t(0) = 0$ and

$$\mathcal{L}_{\Xi_t}\varphi_t = \varphi_0 - \varphi_1.$$

Let $\Xi_t = \sum A_j(x,t)\frac{\partial}{\partial x_j}$. Then our problem becomes: find functions $A_j(x,t)$ with $A_j(0,t) = 0$, such that

$$\sum A_j(x,t)\frac{\partial \varphi_t}{\partial x_j} = \varphi_0 - \varphi_1. \tag{12}$$

According to the second and the third conditions, $[\frac{\partial^2 \varphi_t}{\partial x_i \partial x_j}(0)]$ is non-degenerate. It follows that the system of functions

$$\left\{\frac{\partial \varphi_t}{\partial x_i} \mid i = 1, 2, \cdots, n\right\}$$

form a system of coordinates near 0 with $\frac{\partial \varphi_t}{\partial x_i}(0) = 0$. So according to the next lemma, one can find functions $B_{ij}(x,t)$ so that

$$\varphi_0 - \varphi_1 = \sum B_{ij}(x,t)\frac{\partial \varphi_t}{\partial x_i}\frac{\partial \varphi_t}{\partial x_j}.$$

Obviously if we take $A_j(x,t) = \sum_i B_{ij}(x,t)\frac{\partial \varphi_t}{\partial x_i}$, then $A_j(0,t) = 0$ and satisfies the equation (12). This proves the existence of the diffeomorphism ρ. □

Lemma 2.6. *Let φ be a smooth function with $\varphi(0) = 0$ and $\partial^\alpha \varphi(0) = 0$ for all $|\alpha| < k$. Then there exists smooth functions φ_α so that*

$$\varphi(x) = \sum_{|\alpha|=k} x^\alpha \varphi_\alpha(x). \tag{13}$$

Proof. For $k = 1$, one just take

$$\varphi_j(x) = \int_0^1 \frac{\partial \varphi}{\partial x_j}(x_1, \cdots, x_{j-1}, tx_j, 0, \cdots, 0)dt.$$

For larger k, apply the above formula and induction. □

By exactly the same method, one can prove the following *Morse Lemma with parameters*:

Theorem 2.7 (Morse Lemma with parameters)**.** *Let* $\varphi_i^s \in C^\infty(\mathbb{R}^n)$ *be two family of smooth functions, depending smoothly on the parameter* $s \in \mathbb{R}^k$*. Suppose*

- $\varphi_0^s(0) = \varphi_1^s(0) = 0,$
- $\nabla\varphi_0^s(0) = \nabla\varphi_1^s(0) = 0,$
- $d^2\varphi_0^s(0) = d^2\varphi_1^s(0)$ *are non-degenerate.*

Then there exist an $\varepsilon > 0$*, a neighborhood* U *of* 0 *and for all* $|s| < \varepsilon$ *an open embedding* $\rho_s : U \to \mathbb{R}^n$*, depending smoothly on* s*, so that* $\rho_s(0) = 0$ *and* $\rho_s^*\varphi_1^s = \varphi_0^s$.