# LECTURE 6 — 10/12/2020
# SEMICLASSICAL QUANTIZATION

## 1. The Kohn-Nirenberg quantization

**¶ The semiclassical Fourier transform.**

In semiclassical analysis, instead of the usual Fourier transform, we shall use the *semiclassical Fourier transform* (also known as the *$\hbar$-Fourier transform*) $\mathcal{F}_\hbar$. By definition it is a dilation of the usual Fourier transform,

$$\mathcal{F}_\hbar\varphi(\xi) := (\mathcal{F}\varphi)(\frac{\xi}{\hbar}) = \int_{\mathbb{R}^n} e^{-\frac{ix\cdot\xi}{\hbar}}\varphi(x)dx. \tag{1}$$

In PSet1-6 you are supposed to translate known properties of the usual Fourier transform to the semiclassical Fourier transform. For example, the inverse of $\mathcal{F}_\hbar$ is

$$\mathcal{F}_\hbar^{-1}\psi(x) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{\frac{ix\cdot\xi}{\hbar}}\psi(\xi)d\xi. \tag{2}$$

It follows

$$\psi(0) = [\mathcal{F}_\hbar\mathcal{F}_\hbar^{-1}\psi](0) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}x\cdot\xi}\psi(x)dxd\xi,$$

which can be interpreted as a strange-looking distributional identity

$$\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}x\cdot\xi}d\xi = \delta_0.$$

It is also easy to check

$$\mathcal{F}_\hbar((\hbar D_x)^\alpha\varphi) = \xi^\alpha\mathcal{F}_\hbar\varphi \quad \text{and} \quad \mathcal{F}_\hbar(x^\alpha\varphi) = (-1)^{|\alpha|}(\hbar D_\xi)^\alpha\mathcal{F}_\hbar\varphi.$$

**¶ The Kohn-Nirenberg quantization.**

We have mentioned that a reasonable way to quantize the position and momentum functions is

$$x_j \quad \rightsquigarrow \quad Q_j = \text{ multiplication by } x_j$$

and

$$\xi_j \quad \rightsquigarrow \quad P_j = \frac{\hbar}{\sqrt{-1}}\frac{\partial}{\partial x_j} = \hbar D_j.$$

We want to extend the "same" rule to more general functions. Note that the operator $P_j$ and the function $\xi_j$ are related by the semiclassical Fourier transform, namely

$$P_j\varphi = \hbar D_j\varphi = \mathcal{F}_\hbar^{-1}(\mathcal{F}_\hbar(\hbar D_j)\varphi) = \mathcal{F}_\hbar^{-1}(\xi_j\mathcal{F}_\hbar\varphi).$$

Similarly we have [1]

$$Q_j\varphi = x\varphi = \mathcal{F}_\hbar^{-1}(x\mathcal{F}_\hbar\varphi).$$

More generally, we may replace $Q_j$ by any function $V(x)$ of $x$ to get

$$V(x)\varphi = \mathcal{F}_\hbar^{-1}(V(x)\mathcal{F}_\hbar\varphi),$$

that is, the operator "multiplication by $V(x)$" and the function $V(x)$ are related by the same formula. Similarly we can relate the polynomial

$$p(\xi) = \sum_{|\alpha|\le k} p_\alpha \xi^\alpha$$

with the the constant coefficient semiclassical differential operator $p(\hbar D)$ defined by

$$p(\hbar D) = \sum_{|\alpha|\le k} p_\alpha (\hbar D_x)^\alpha$$

via exactly the same rule:

$$p(\hbar D)\varphi = \mathcal{F}_\hbar^{-1}(\mathcal{F}_\hbar(p(\hbar D)\varphi)) = \mathcal{F}_\hbar^{-1}(p(\xi)\mathcal{F}_\hbar\varphi).$$

Bingo! In particular, by using $\hbar$-Fourier transform we get an explanation of

$$H(x,\xi) = \frac{|\xi|^2}{2} + V(x) \quad \rightsquigarrow \quad \hat{H} = -\frac{\hbar^2}{2}\Delta + V(x),$$

since the Hamiltonian function $H$ and the Schrödinger operator $\hat{H}$ are related by

$$\hat{H}\varphi = \mathcal{F}_\hbar^{-1}(H(x,\xi)\mathcal{F}_\hbar\varphi).$$

We can go further. Still suppose $p$ is a polynomial in $\xi$, but now with coefficients depending on $x$, namely

$$p(x,\xi) = \sum_{|\alpha|\le k} p_\alpha(x)\xi^\alpha.$$

Then we can do the same computation

$$\mathcal{F}_\hbar^{-1}(\sum_{|\alpha|\le k} p_\alpha(x)\xi^\alpha \mathcal{F}_\hbar\varphi) = \sum_{|\alpha|\le k} p_\alpha(x)\mathcal{F}_\hbar^{-1}\mathcal{F}_\hbar((\hbar D)^\alpha\varphi) = \sum_{|\alpha|\le k} p_\alpha(x)(\hbar D_x)^\alpha$$

and thus we arrive at the *semiclassical differential operator* of the form

$$p(x,\hbar D) = \sum p_\alpha(x)(\hbar D_x)^\alpha.$$

We can apply the same construction to many other classes of functions. For simplicity, let's first suppose $a(x,\xi) \in \mathcal{S}(\mathbb{R}^{2n})$ is a Schwartz function (which is the *best* class of functions), we may quantize $a$ to the operator $\hat{a}^{KN}$ given by

$$\varphi \mapsto \hat{a}^{KN}(\varphi) := \mathcal{F}_\hbar^{-1}(a(x,\xi)\mathcal{F}_\hbar\varphi) = (\mathcal{F}_\hbar^{-1})_{\xi\to x}(a(x,\xi)(\mathcal{F}_\hbar)_{y\to\xi}(\varphi(y))).$$

[1] Here and in what follows, we can "move" a function depending only on $x$ out of $\mathcal{F}_\hbar^{-1}$ because the inverse Fourier transform $\mathcal{F}_\hbar^{-1}$ is an integral with respect to $\xi$. *Warning: although we have both $x\varphi = \mathcal{F}_\hbar^{-1}(x\mathcal{F}_\hbar\varphi)$ and $x\varphi = \mathcal{F}_\hbar^{-1}(\mathcal{F}_\hbar(x\varphi))$, we can't conclude $x\mathcal{F}_\hbar\varphi = \mathcal{F}_\hbar(x\varphi)$, since $x\mathcal{F}_\hbar\varphi$ is a function depending on both $x$ and $\xi$, while $\mathcal{F}_\hbar(x\varphi)$ is a function only depending on $\xi$.*

The quantization process

$$a \rightsquigarrow \hat{a}^{KN}$$

is called the *Kohn-Nirenberg quantization* (also known as the *standard quantization*). By using the definition of the semiclassical Fourier and inverse Fourier transforms, we can easily write down an explicit formula for $\hat{a}^{KN}$:

$$\begin{aligned}\hat{a}^{KN}(\varphi) &= \mathcal{F}_\hbar^{-1}(a(x,\xi)\mathcal{F}_\hbar\varphi)\\ &= \mathcal{F}_\hbar^{-1}\left(a(x,\xi)\int_{\mathbb{R}^n} e^{-i\frac{y\cdot\xi}{\hbar}}\varphi(y)dy\right)\\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}}a(x,\xi)\varphi(y)dyd\xi.\end{aligned}$$

**¶ DETOUR: Schwartz kernel of an integral operator.**

Note that if we denote

$$k_a^{KN}(x,y) := \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}}a(x,\xi)d\xi,$$

then the expression of $\hat{a}^{KN}$ can be simplified to

$$\hat{a}^{KN}(\varphi)(x) = \int_{\mathbb{R}^n} k_a^{KN}(x,y)\varphi(y)dy.$$

In general, given any reasonable (e.g. smooth or measurable, bounded or integrable etc.)*kernel function* $k(x,y)$, where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, we can define a linear operator $K$ which maps a function on $\mathbb{R}^n$ to a function on $\mathbb{R}^m$ via the integral

$$K(\varphi)(x) = \int_{\mathbb{R}^n} k(x,y)\varphi(y)dy. \tag{3}$$

Of course the domain of the integral operator $K$ depends on the function $k(x,y)$: nicer kernel function usually admits larger domain. Here we discuss two extremal cases:

- Case 1: The best kernels, namely $k \in \mathscr{S}(\mathbb{R}^m \times \mathbb{R}^n)$.
  In this case, for the integral (3) to make sense, you may take $\varphi$ to be bounded continuous functions, or if you want, $L^p$ functions. In fact, you can go further:

  Fact: We may take $\varphi$ to be a tempered distribution!

  Of course in this case one has to interpret the integral (3) as a pairing between tempered distributions and Schwartz functions: for each $x$ fixed, the function $k(x,\cdot)$ is a Schwartz function, and we define

$$K(\varphi)(x) := \langle \varphi, k(x,\cdot)\rangle. \tag{4}$$

  A natural question to ask is: what do we get? Is $K(\varphi)$ a nice function or a bad function? Or maybe only a (very bad) tempered distribution? The answer is:

  Fact: $K$ maps a tempered distribution to a Schwartz function!

  To see this, we use the following remarkable theorem in distribution theory:

**Theorem 1.1** (Schwartz representation theorem). *For any $u \in \mathscr{S}'(\mathbb{R}^n)$, there exists a finite collection $u_{\alpha,\beta} : \mathbb{R}^n \to \mathbb{C}$ of bounded continuous functions, with $|\alpha| + |\beta| \le k$, such that*

$$u = \sum_{|\alpha|+|\beta|\le k} x^\alpha D_x^\beta u_{\alpha,\beta}.$$

Note that both sides of the equation above are understood as distributions: In Lecture 4 we have seen how to realize any bounded continuous function as a tempered distribution, and how to realize $x^\alpha D^\beta u$ as a distribution when $u$ is a distribution. As a consequence, for a tempered distribution $\varphi \in \mathscr{S}'(\mathbb{R}^n)$, the "paring formula" (4) of $K(\varphi)$ still has an integral representation

$$\begin{aligned} K(\varphi)(x) &= \left\langle \sum_{|\alpha|+|\beta|\le k} y^\alpha D_y^\beta \varphi_{\alpha,\beta}(y), k(x,y) \right\rangle \\ &= \sum_{|\alpha|+|\beta|\le k} \left\langle \varphi_{\alpha,\beta}(y), D_y^\beta y^\alpha k(x,y) \right\rangle \\ &= \sum_{|\alpha|+|\beta|\le k} \int_{\mathbb{R}^n} \varphi_{\alpha,\beta}(y) D_y^\beta \left( y^\alpha k(x,y) \right) dy \end{aligned}$$

and thus by using the fact $k \in \mathscr{S}(\mathbb{R}^m \times \mathbb{R}^n)$, one can check $K(\varphi) \in \mathscr{S}(\mathbb{R}^m)$. Moreover, it can be shown that the operator $K : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^m)$ is continuous (with respect to the weak-$*$ topology on $\mathscr{S}'(\mathbb{R}^n)$ and the metric topology on $\mathscr{S}(\mathbb{R}^m)$). In summary, we have

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|---|
| If the kernel function $k \in \mathscr{S}(\mathbb{R}^m \times \mathbb{R}^n)$ is a Schwartz function, then the integral operator $K$ with kernel $k$ can be defined as a continuous linear operator mapping $\mathscr{S}'(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^m)$. |

- Case 2: The worst kernels, namely $k \in \mathscr{S}'(\mathbb{R}^m \times \mathbb{R}^n)$.
To give an exact meaning of the operator $K$ with a distributional kernel $k$, we first introduce a notation: Given any $\varphi \in C^\infty(\mathbb{R}^n)$ and $\psi \in C^\infty(\mathbb{R}^m)$, we define $\varphi \boxtimes \psi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ to be the function given by

$$\varphi \boxtimes \psi(x,y) := \varphi(x)\psi(y).$$

Obviously if $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and $\psi \in \mathscr{S}(\mathbb{R}^m)$, then $\varphi \boxtimes \psi \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^m)$.

Now suppose $k \in \mathscr{S}'(\mathbb{R}^m \times \mathbb{R}^n)$ is a tempered distribution. We can still define an operator $K : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^m)$ via the formula[2]

$$K(\varphi)(\psi) := \langle k, \varphi \boxtimes \psi \rangle.$$

and one can show that $K$ is a continuous linear map. In fact, it turns out that any continuous linear map $\mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^m)$ arises in this way:

[2] In view of the Schwartz representation theorem, the operator $K$ is really an integral operator.

**Theorem 1.2** (Schwartz kernel theorem). *There is a one-to-one correspondence between continuous linear operators $K : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^m)$ and their kernels $k \in \mathscr{S}'(\mathbb{R}^m \times \mathbb{R}^n)$.*

So usually we will call the kernel function $k$ the *Schwartz kernel* of the integral operator $K$.

*Remark.* As we have mentioned, in this course the main objects are pseudodifferential operator sand Fourier integral operators. They are all integral operators defined via Schwartz kernels. In what follows, when we write down such an integral expression in which there could be some convergence issue, we will explain the expression in the sense of distribution.

*Remark.* Integral operators with Schwartz functions as Schwartz kernels are very nice (we will prove many other nice properties of such operators later), but they are too restrictive: for example, the Schwartz kernel of semiclassical differential operators are polynomials in $\xi$ which are not Schwartz functions. On the other hand, integral operators with distributional kernel contains all possible integral operators, but usually they don't have nice properties: for example, in general we can't composite two such operators.

Back to the Kohn-Nirenberg quantization. Now we can say: not only we can quantize Schwartz functions or polynomials, but also we can quantize tempered distributions $a \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n)$, in which case the operator $\hat{a}^{KN}$ is the integral operator whose Schwartz kernel is the tempered distribution

$$\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} a(x,\xi)d\xi = (\mathcal{F}_\hbar^{-1})_{\xi\to x-y}(a(x,\xi)).$$

We will introduce many other classes of functions (called *symbol classes*) so that the resulting classes of integral operators are large enough <u>AND</u> nice enough.

## 2. Other semiclassical quantizations

**¶ The anti-Kohn-Nirenberg quantization.**

Back to semiclassical quantization. By our construction, the Kohn-Nirenberg quantization quantize the function $a(x,\xi) = x_1\xi_1$ to the operator $\hat{a}^{KN} = Q_1P_1$. However, in Lecture 1 we have already mentioned that since

$$x_1\xi_1 = \xi_1x_1 = (x_1\xi_1 + \xi_1x_1)/2,$$

instead of quantize $x_1\xi_1$ to $Q_1P_1$ one could also quantize it to $P_1Q_1$ or even $(Q_1P_1 + P_1Q_1)/2$. It turns out that there do exist very similar theories, again using the semiclassical Fourier transform, which quantize $x_1\xi_1$ to $P_1Q_1$ to $(Q_1P_1 + P_1Q_1)/2$.

To see how to get the operator $P_1Q_1$ out of the function $x_1\xi_1$, let's do the same computation as before:

$$P_1Q_1\varphi = \hbar D_{x_1}(x_1\varphi) = \mathcal{F}_\hbar^{-1}\mathcal{F}_\hbar(\hbar D_{x_1}(x_1\varphi)) = \mathcal{F}_\hbar^{-1}(\xi_1\mathcal{F}_\hbar(x_1\varphi)) = \mathcal{F}_\hbar^{-1}(\mathcal{F}_\hbar(\xi_1x_1\varphi)).$$

We emphasis again that in the last expression, we can't eliminate $\mathcal{F}_\hbar^{-1}$ with $\mathcal{F}_\hbar$ since the Fourier transform is acting on a function that depends on both $\xi$ and $x$. Since we usually write the resulting function as a function in $x$, To avoid possible misunderstanding, let's rewrite the expression above as

$$(P_1Q_1\varphi)(x) = (\mathcal{F}_\hbar^{-1})_{\xi\to x}(\mathcal{F}_\hbar)_{y\to\xi}(y_1\xi_1\varphi(y)).$$

In general, given more complicated functions $a(x,\xi)$, we may quantize it to the integral operator $\hat{a}^{anti-KN}$ given by

$$(\hat{a}^{anti-KN}\varphi)(x) := (\mathcal{F}_\hbar^{-1})_{\xi\to x}(\mathcal{F}_\hbar)_{y\to\xi}(a(y,\xi)\varphi(y))$$

Such a quantization is called the *anti-Kohn-Nirenberg* quantization. For example, if we take $a(x,\xi)$ to be the polynomial

$$p(x,\xi) = \sum_{|\alpha|\le k} p_\alpha(x)\xi^\alpha,$$

then a simple computation will yield (do this computation by yourself)

$$\hat{p}^{anti-KN}\varphi(x) = \sum_{|\alpha|\le k} D^\alpha(p_\alpha(x)\varphi(x)).$$

So again we quantize polynomials to semiclassical differential operators. However, unlike the Kohn-Nirenberg quantization in which we get differential operators with $x$'s before $D$'s, now we get differential operators with $D$'s before $x$'s.

*Remark.* Some authors prefer to call the Kohn-Nirenberg quantization "left quantization" and the anti-Kohn-Nirenberg quantization "right quantization", while some other authors prefer to use the opposite, namely, call the Kohn-Nirenberg quantization "right quantization" and the anti-Kohn-Nirenberg quantization "left quantization".

It is not hard to write down an explicit formula for the anti-Kohn-Nirenberg quantization:

$$\begin{aligned}(\hat{a}^{anti-KN}\varphi)(x) &= (\mathcal{F}_\hbar^{-1})_{\xi\to x}(\mathcal{F}_\hbar)_{y\to\xi}(a(y,\xi)\varphi(y))\\ &= (\mathcal{F}_\hbar^{-1})_{\xi\to x}\left(\int_{\mathbb{R}^n} e^{-i\frac{y\cdot\xi}{\hbar}}a(y,\xi)\varphi(y)dy\right)\\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}}a(y,\xi)\varphi(y)dyd\xi.\end{aligned}$$

As in the Kohn-Nirenberg case, the anti-Kohn-Nirenberg quantization of $a$ is an integral operator with Schwart kernel

$$k_a^{anti-KN}(x,y) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}}a(y,\xi)d\xi = (\mathcal{F}_\hbar^{-1})_{\xi\to x-y}(a(y,\xi))$$

which makes sense even if $a$ is only a tempered distribution.

**¶ The Weyl quantization.**

As we have seen, the Kohn-Nirenberg quantization "prefer" to put $D$ after $x$ while the anti-Kohn-Nirenberg quantization "prefer" to put $D$ before $x$. But why should we choose such an order when we do quantization? At least from the classical mechanics point of view, the position function and the momentum function should behave equally. There is a more natural way to quantize, the *Weyl quantization*[3], in which we quantize $x_1\xi_1$ to the more "balanced" operator $(Q_1P_1+P_1Q_1)/2$. As we will see, when compared with the Kohn-Nirenberg or anti-Kohn-Nirenberg quantizations, the Weyl quantization has many nice properties (but in some situations the later are easier to do computations).

To write down an explicit formula for the Weyl quantization, let's compute:

$$\begin{aligned}\frac{Q_1P_1+P_1Q_1}{2}\varphi &= \frac{\widehat{x_1\xi_1}^{KN}+\widehat{x_1\xi_1}^{anti-KN}}{2}\varphi\\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}}\frac{x_1+y_1}{2}\xi_1\varphi(y)dyd\xi.\end{aligned}$$

Note that for $a(x,\xi)=x_1\xi_1$, we have $a(\frac{x+y}{2},\xi)=\frac{x_1+y_1}{2}\xi_1$. So in general we define the *Weyl quantization* of $a(x,\xi)$ to be the linear operator $\hat{a}^W$ given by

$$(\hat{a}^W\varphi)(x)=\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}}a(\frac{x+y}{2},\xi)\varphi(y)dyd\xi.$$

Again this is an integral operator with Schwartz kernel

$$k_a^W(x,y)=\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}}a(\frac{x+y}{2},\xi)d\xi=(\mathcal{F}_\hbar^{-1})_{\xi\to x-y}(a(\frac{x+y}{2},\xi))$$

which makes sense even if $a$ is only a tempered distribution.

**¶ Semiclassical $t$-quantizations.**

It is possible to unite the Kohn-Nirenberg quantization, the anti-Kohn-Nirenberg quantization and the Weyl quantization in one formula:

**Definition 2.1.** For any $0\le t\le 1$ we define the *semiclassical $t$-quantization* (also known as the *Shubin $t$-quantization*) of $a$ to be the operator $Op_\hbar^t(a)$ given by

$$Op_\hbar^t(a)(\varphi)(x)=\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}}a(tx+(1-t)y,\xi)\varphi(y)dyd\xi. \tag{5}$$

Any operator of the form (5) is called a *semiclassical pseudo-differential operator*, or a *$\hbar$-pseudo-differential operator*. The function $a$ is called the *$t$-symbol* of $Op_\hbar^t(a)$.

Obviously the cases $t=1,0,\frac12$ are exactly the Kohn-Nirenberg quantization, the anti-Kohn-Nirenberg quantization and the Weyl quantization that we just studied.

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[3]Historically, the Weyl quantization appears much earlier than the Kohn-Nirenberg or the anti-Kohn-Nirenberg quantizations.

*Remark.* It is easy to see that for any $0 \le t \le 1$,

- the operator $Op_\hbar^t(a)$ is an integral operator with Schwartz kernel
$$k_a^t(x,y) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} a(tx+(1-t)y,\xi)d\xi.$$
- if $a$ is a polynomial in $\xi$, then $Op_\hbar^t(a)$ is a semiclassical differential operator.
- if $a \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$, then the operator $Op_\hbar^t(a)$ maps $\mathscr{S}'(\mathbb{R}^n)$ continuously into $\mathscr{S}(\mathbb{R}^n)$.
- if $a \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n)$, then the operator $Op_\hbar^t(a)$ maps $\mathscr{S}(\mathbb{R}^n)$ continuously into $\mathscr{S}'(\mathbb{R}^n)$.

*Remark.* More generally, one can study integral operators of the same form but with a general amplitude $a(x,y,\xi)$. We will also allow the amplitude $a$ to be $\hbar$-dependent. However, it turns out that the class of operators defined via such general amplitudes coincide with the class of operators defined via $t$-quantization (with $\hbar$-dependent symbol) for any fixed $t$.

**¶ Formal adjoint.**

By definition, the complex conjugate of $k_a^t(x,y)$ is
$$\overline{k_a^t(y,x)} = k_{\bar a}^{1-t}(x,y).$$

It follows

**Lemma 2.2.** *The operator $Op_\hbar^{1-t}(\bar a)$ is the formal adjoint of $Op_\hbar^t(a)$.*

*Proof.* By definition, for $\varphi,\psi \in \mathscr{S}(\mathbb{R}^n)$,
$$\begin{aligned}
\langle Op_\hbar^t(a)\varphi,\psi\rangle &= \int_{\mathbb{R}^n}(Op_\hbar^t(a)\varphi)(x)\overline{\psi(x)}dx\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k_a^t(x,y)\varphi(y)\overline{\psi(x)}dydx\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(x)k_a^t(y,x)\overline{\psi(y)}dxdy\\
&= \int_{\mathbb{R}^n}\varphi(x)\overline{\int_{\mathbb{R}^n}\overline{k_a^t(y,x)}\psi(y)dy}dx\\
&= \langle\varphi, Op_\hbar^{1-t}(\bar a)\varphi\rangle.
\end{aligned}$$

□

As we have explained in Lecture 2, we would like to quantize a real-valued function to a self-adjoint operator (whose spectrum are real numbers). This is the second evidence that the Weyl quantization is more natural:

**Corollary 2.3.** *If $a$ is real-valued, then the Weyl quantization $\hat a^W$ is formally self-adjoint.*

We will see that $\hat a^W$ is bounded on $L^2$ for a very large class of symbols, in which case $a^W$ is self-adjoint if $a$ is real.