LECTURE 7 — 10/14/2020
WEYL QUANTIZATION: EXAMPLES

1. Weyl quantization of polynomial-type functions

Today we focus on the Weyl quantization

$$\widehat{a}^W(\varphi)(x) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} a(\frac{x+y}{2},\xi)\varphi(y)dyd\xi.$$

We shall compute the operator $\widehat{a}^W$ for some simple classes of functions. A formula that we will use several times is the Fourier inversion formula,

$$f(x) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} f(y)dyd\xi,$$

or more precisely, its variation

$$f(x,x) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} f(x,y)dyd\xi,$$

which can be obtained from the following identity by setting $u = x$:.

$$f(u,x) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} f(u,y)dyd\xi.$$

¶ Weyl quantization of $a(x)$.

We start with the case $\boxed{a = a(x)}$:

$$\widehat{a}^W(\varphi)(x) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} a(\frac{x+y}{2})\varphi(y)dyd\xi = a(x)\varphi(x).$$

So, as one can expect (which holds for all t-quantizations)

$$\widehat{a(x)}^W = \text{“multiplication by } a(x)\text{”}.$$

¶ Weyl quantization of $a(\xi)$.

Next let's consider the case $\boxed{a = a(\xi)}$:

$$\begin{aligned}\widehat{a}^W(\varphi)(x) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} a(\xi)\varphi(y)dyd\xi \\ &= (\mathcal{F}_\hbar^{-1})_{\xi\to x}\left[a(\xi)(\mathcal{F}_\hbar\varphi)(\xi)\right](x).\end{aligned}$$

So we get (again the same formula holds (trivially) for all t-quantizations):

$$\widehat{a(\xi)}^W = \mathcal{F}_\hbar^{-1}\circ a(\xi)\circ\mathcal{F}_\hbar.$$

Such operators are known as *Fourier multipliers*. Note that in particular we get $\widehat{\xi^\alpha}^W = P^\alpha$, and thus $(\widehat{|\xi|^2/2 + V(x))}^W = -\hbar^2\Delta/2 + V$.

As an application, we calculate the Weyl quantization of the quadratic exponential $\boxed{a(\xi) = e^{\frac{i}{2\hbar}\xi^T Q\xi}}$, where Q is a non-singular symmetric $n\times n$ matrix. Denote

$$p_Q(\xi) = \xi^T Q\xi.$$

Proposition 1.1. *For* $a(\xi) = e^{\frac{i}{2\hbar}\xi^T Q\xi}$ *we have*

$$\hat{a}^W\varphi(x) = \frac{|\det Q|^{-1/2}}{(2\pi\hbar)^{n/2}} e^{i\frac{\pi}{4}\mathrm{sgn}Q}\int_{\mathbb{R}^n} e^{-\frac{i}{2\hbar}y^T Q^{-1}y}\varphi(x+y)dy. \tag{1}$$

Proof. In Lecture 4 we showed

$$\mathcal{F}(e^{\frac{i}{2}x^T Qx}) = \frac{(2\pi)^{n/2}e^{i\frac{\pi}{4}\mathrm{sgn}(Q)}}{|\det Q|^{\frac{1}{2}}}e^{-\frac{i}{2}\xi^T Q^{-1}\xi}.$$

It follows

$$\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}x\cdot\xi}e^{\frac{i}{2\hbar}\xi^T Q\xi}d\xi = \mathcal{F}_\hbar^{-1}(e^{\frac{i}{2\hbar}p_Q(\xi)})(x) = \frac{|\det Q|^{-1/2}}{(2\pi\hbar)^{n/2}}e^{i\frac{\pi}{4}\mathrm{sgn}Q}e^{-\frac{i}{2\hbar}p_{Q^{-1}}(x)}.$$

Thus

$$\begin{aligned}\hat{a}^W\varphi(x) =& \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi}e^{\frac{i}{2\hbar}\xi^T Q\xi}\varphi(y)dyd\xi \\ =& \frac{|\det Q|^{-1/2}}{(2\pi\hbar)^{n/2}}e^{i\frac{\pi}{4}\mathrm{sgn}Q}\int_{\mathbb{R}^n} e^{-\frac{i}{2\hbar}p_{Q^{-1}}(x-y)}\varphi(y)dy \\ =& \frac{|\det Q|^{-1/2}}{(2\pi\hbar)^{n/2}}e^{i\frac{\pi}{4}\mathrm{sgn}Q}\int_{\mathbb{R}^n} e^{-\frac{i}{2\hbar}p_{Q^{-1}}(y)}\varphi(x+y)dy.\end{aligned}$$

□

The formula (1) will be used next time to compute the symbol of the composition of two Weyl operators.

¶ Weyl quantization of polynomials in both x and ξ.

By linearity, to compute the Weyl quantization of a polynomial in both x and ξ, it is enough to compute the Weyl quantization of monomials $\boxed{a(x,\xi) = x^\alpha\xi^\beta}$:

$$\begin{aligned}\hat{a}^W(\varphi)(x) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}}(\frac{x+y}{2})^\alpha\xi^\beta\varphi(y)dyd\xi \\ &= \sum_{\gamma\le\alpha}\frac{1}{2^{|\alpha|}}\binom{\alpha}{\gamma}x^\gamma\widehat{x^{\alpha-\gamma}\xi^\beta}^{anti-KN}(\varphi)(x) \\ &= \sum_{\gamma\le\alpha}\frac{1}{2^{|\alpha|}}\binom{\alpha}{\gamma}x^\gamma(\hbar D_x)^\beta(x^{\alpha-\gamma}\varphi(x)).\end{aligned}$$

In other words, we get the following *McCoy's formula*:

$$\widehat{x^\alpha \xi^\beta}^W = \sum_{\gamma \le \alpha} \frac{1}{2^{|\alpha|}} \binom{\alpha}{\gamma} Q^\gamma P^\beta Q^{\alpha-\gamma},$$

where $\gamma \le \alpha$ means $\gamma_j \le \alpha_j$ for all j, and

$$\binom{\alpha}{\gamma} := \frac{\alpha!}{\gamma!(\alpha-\gamma)!} = \binom{\alpha_1}{\gamma_1} \cdots \binom{\alpha_n}{\gamma_n}.$$

¶ Weyl quantization of polynomials in ξ.

Next let's compute the Weyl quantization of $\boxed{a(x,\xi) = \sum_{|\alpha| \le k} a_\alpha(x) \xi^\alpha}$. Using the fact

$$(-\hbar D_y)^\alpha e^{i\frac{(x-y)\cdot\xi}{\hbar}} = \xi^\alpha e^{i\frac{(x-y)\cdot\xi}{\hbar}}$$

and the Fourier inversion formula we get

$$\begin{aligned}
\hat{a}^W(\varphi)(x) &= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} \sum_{|\alpha|\le k} a_\alpha(\frac{x+y}{2}) \xi^\alpha \varphi(y) dy d\xi \\
&= \sum_{|\alpha|\le k} \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left((-\hbar D_y)^\alpha e^{i\frac{(x-y)\cdot\xi}{\hbar}} \right) a_\alpha(\frac{x+y}{2}) \varphi(y) dy d\xi \\
&= \sum_{|\alpha|\le k} \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} (\hbar D_y)^\alpha \left(a_\alpha(\frac{x+y}{2}) \varphi(y) \right) dy d\xi \\
&= \sum_{|\alpha|\le k} (\hbar D_y)^\alpha \left[a_\alpha(\frac{x+y}{2}) \varphi(y) \right]_{y=x} \\
&= \sum_{|\alpha|\le k} \sum_{\gamma\le\alpha} 2^{-|\gamma|} \binom{\alpha}{\gamma} [(\hbar D)^\gamma a_\alpha(x)] \cdot (\hbar D)^{\alpha-\gamma} \varphi(x).
\end{aligned}$$

So we get

$$\hat{a}^W = \sum_{|\alpha|\le k} \sum_{\gamma\le\alpha} 2^{-|\gamma|} \binom{\alpha}{\gamma} [(\hbar D)^\gamma a_\alpha(x)] \cdot (\hbar D)^{\alpha-\gamma}.$$

As a consequence

Corollary 1.2. *If $a(x,\xi) = \sum_{|\alpha|\le k} a_\alpha(x)\xi^\alpha$ is a polynomial of degree k in ξ, then $\hat{a}^W$ is a semiclassical differential operator of order k of the form*

$$\hat{a}^W = \sum_{|\alpha|=k} a_\alpha(x)(\hbar D)^\alpha + \text{“terms of order } \le k-1\text{”}.$$

Note that the same result holds for the Kohn-Nirenberg quantization and the anti-Kohn-Nirenberg quantization, and in fact for all semiclassical t-quantizations.

2. Symplectic invariance and applications

¶ Symplectic invariance of Weyl quantization.

According to the computations above, it seems that Weyl quantization is much more complicated than the Kohn-Nirenberg or the anti-Kohn-Nirenberg quantizations. A natural question is: what is the advantage of the Weyl quantization? We have seen the first big advantage: Weyl quantization will quantize real-valued functions to formally self-adjoint operators. Here we explain the second big advantage: the (unitary) invariance under linear symplectomorphisms [1](this conception will be explained later).

Theorem 2.1 (Symplectic invariance of Weyl quantization). *Given any "linear symplectomorphism" $\Phi : T^*\mathbb{R}^n \to T^*R^n$, there is a* metaplectic operator U_Φ *(which is an isomorphism on $\mathscr{S}'(\mathbb{R}^n)$ and on $\mathscr{S}(\mathbb{R}^n)$, and is unitary on $L^2(\mathbb{R}^n)$) such that*

$$\widehat{a \circ \Phi}^W = U_\Phi^{-1} \circ \widehat{a}^W \circ U_\Phi. \tag{2}$$

Remark. Moreover, it can be shown that such "symplectic invariance" characterize the Weyl quantization: If there is a "quantization process" $Q : \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{L}(\mathscr{S}(\mathbb{R}^n), \mathscr{S}'(\mathbb{R}^n))$ which is sequentially continuous, quantizes any bounded function $a(x)$ to the operator "multiplication by $a(x)$" and satisfies the symplectic invariance property above, then it is the Weyl quantization!

Here are three special cases of this theorem for which we can easily check (2) by direct computations:

(A) The linear symplectomorphism is the map

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad (x, \xi) \mapsto (\xi, -x)$$

which "intertwines" x and ξ with a twisting. In this case $U_\Phi = \mathcal{F}_\hbar$. (So $\mathcal{F}_\hbar$ can be regarded as the quantization of J!)

(B) The linear symplectomorphism is the map

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad (x, \xi) \mapsto (x, \xi + Cx),$$

where C is a symmetric $n \times n$ matrix. In this case U_Φ is the map "multiplication by the function $e^{ix^TCx/2\hbar}$".

(C) The linear symplectomorphism is the map

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad (x, \xi) \mapsto (Ax, (A^T)^{-1}\xi),$$

where A is an invertible matrix. In this case U_Φ is the map U_Φ is given by $(U_\Phi\varphi)(x) = \varphi(Ax)$.

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[1]From the classical-quantum correspondence point of view, a nice quantization should preserve symplectic properties, two symplectically equivalent classical objects should corresponds to unitarily equivalent quantum objects. In Lecture 1 we have mentioned the Egorov theorem, which can be explained as the unitary invariance under general symplectomorphisms (which only hold in the semiclassical limit). Here, for linear symplectomorphisms, the invariance is an exact relation.

Remark. In fact, one can prove that any linear symplectomorphism can be written as a composition of the three classes of linear symplectomorphisms above. As a result, the general theorem is proven as long as we can check the three cases (A), (B) and (C). We will not prove the full theorem here [2]. Instead, in what follows we will prove case (A) and a special case of (B), and give two applications. We will leave the proof of the general cases of (B) and (C) as an exercise.

¶ Case (A): Conjugation by Fourier transform.

We prove case (A) by direct computation:

Theorem 2.2 (Conjugation by Fourier transform)**.** *Let* $b(x,\xi) = a(\xi, -x)$*, then*

$$\mathcal{F}_\hbar^{-1} \circ \widehat{a}^W \circ \mathcal{F}_\hbar = \widehat{b}^W. \tag{3}$$

Proof. We compute

$$\begin{aligned}
&\left[(\mathcal{F}_\hbar^{-1})_{\eta\to x}(\widehat{a}^W)_{y\to\eta}(\mathcal{F}_\hbar)_{z\to y}\varphi\right](x)\\
=&(\mathcal{F}_\hbar^{-1})_{\eta\to x}(\widehat{a}^W)_{y\to\eta}\left[\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}z\cdot y}\varphi(z)dz\right]\\
=&(\mathcal{F}_\hbar^{-1})_{\eta\to x}\left[\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(\eta-y)\cdot\xi}a(\frac{\eta+y}{2},\xi)e^{-\frac{i}{\hbar}z\cdot y}\varphi(z)dzdyd\xi\right]\\
=&\frac{1}{(2\pi\hbar)^n}\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}x\cdot\eta}e^{\frac{i}{\hbar}(\eta-y)\cdot\xi}a(\frac{\eta+y}{2},\xi)e^{-\frac{i}{\hbar}z\cdot y}\varphi(z)dzdyd\xi d\eta\\
=&\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\left[\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}[x\cdot\eta+(\eta-z)\cdot\xi-y\cdot z]}a(\frac{\eta+z}{2},\xi)dzd\xi d\eta\right]\varphi(y)dy\\
=&\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\left[\frac{2^n}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}[x\cdot\eta+2(\eta-\zeta)\cdot\xi-y\cdot(2\zeta-\eta)]}a(\zeta,\xi)d\zeta d\xi d\eta\right]\varphi(y)dy\\
=&\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\left[\frac{2^n}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x+y+2\xi)\cdot\eta}e^{\frac{i}{\hbar}(-2\xi-2y)\cdot\zeta}a(\zeta,\xi)d\zeta d\xi d\eta\right]\varphi(y)dy\\
=&\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\left(\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}\tau\cdot\eta}e^{\frac{i}{\hbar}(x-y-\tau)\cdot\zeta}a(\zeta,\frac{\tau-x-y}{2})d\tau d\eta\right)d\zeta\right]\varphi(y)dy
\end{aligned}$$

Using the Fourier inversion formula

$$f(0) = [\mathcal{F}_\hbar\mathcal{F}_\hbar^{-1}f](0) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}x\cdot\xi}f(x)dxd\xi,$$

the expression in $(\cdots)$ above can be simplified to

$$e^{\frac{i}{\hbar}(x-y)\cdot\zeta}a(\zeta,\frac{-x-y}{2}) = e^{\frac{i}{\hbar}(x-y)\cdot\zeta}b(\frac{x+y}{2},\zeta)$$

and the conclusion follows. □

[2] For a proof, c.f. Folland, *Harmonic analysis in phase space*, Chapter 4.

¶ Application: A second formula for the Weyl quantization of polynomials in ξ.

As an application, to compute the Weyl quantization of $a(x,\xi)=\sum_{|\alpha|\le k} a_\alpha \xi^\alpha$, we can intertwine x and ξ first, so that instead of handling the function $a(\frac{x+y}{2})$, we only need to handle the polynomial $(\frac{x+y}{2})^\alpha$ by using the binomial theorem:

Proposition 2.3. *The Weyl quantization of $a(x,\xi)=\sum_{|\alpha|\le k} a_\alpha \xi^\alpha$ is*

$$\widehat{a}^W = \sum_{|\alpha|\le k}\sum_{\gamma\le\alpha} 2^{-|\alpha|}\binom{\alpha}{\gamma}(\hbar D)^\gamma \circ a_\alpha(x)\circ(\hbar D)^{\alpha-\gamma} \tag{4}$$

Proof. We will apply the previous theorem to "intertwine x and ξ". For this purpose we let $b(x,\xi)=\sum a_\alpha(-\xi)x^\alpha$, then $a(x,\xi)=b(\xi,-x)$ and thus we have

$$\widehat{a}^W = \mathcal{F}_\hbar^{-1}\circ \widehat{b}^W \circ \mathcal{F}_\hbar.$$

Note that by definition and the binomial theorem,

$$\begin{aligned}(\widehat{b}^W\varphi)(x) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi}\sum_{|\alpha|\le k} a_\alpha(-\xi)\left(\frac{x+y}{2}\right)^\alpha \varphi(y)dyd\xi \\ &= \sum_{|\alpha|\le k}\sum_{\gamma\le\alpha} 2^{-|\alpha|}\binom{\alpha}{\gamma}\frac{x^\gamma}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi}a_\alpha(-\xi)y^{\alpha-\gamma}\varphi(y)dyd\xi.\end{aligned}$$

So to prove (4), it remains to check

$$\left[\mathcal{F}_\hbar\circ(\hbar D)^\gamma\circ a_\alpha(x)\circ(\hbar D)^{\alpha-\gamma}\circ\mathcal{F}_\hbar^{-1}\varphi\right](x) = \frac{x^\gamma}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi}a_\alpha(-\xi)y^{\alpha-\gamma}\varphi(y)dyd\xi.$$

This follows from direct computations: The left hand side of the above expression should be interpreted as

$$(\mathcal{F}_\hbar)_{\xi\to x}\circ(\hbar D)_\xi^\gamma\circ a_\alpha(\xi)\circ(\hbar D)_\xi^{\alpha-\gamma}\circ(\mathcal{F}_\hbar^{-1})_{y\to\xi}\varphi(y),$$

which, by using the property $(\mathcal{F}_\hbar)_{\xi\to x}\circ(\hbar D)_\xi^\gamma = x^\gamma(\mathcal{F}_\hbar)_{\xi\to x}$, equals

$$x^\gamma\circ(\mathcal{F}_\hbar)_{\xi\to x}\circ a_\alpha(\xi)\circ(\mathcal{F}_\hbar^{-1})_{y\to\xi}(y^{\alpha-\gamma}\varphi(y)),$$

which equals

$$\begin{aligned}&\frac{x^\gamma}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(y-x)\cdot\xi}a_\alpha(\xi)y^{\alpha-\gamma}\varphi(y)dyd\xi \\ =&\frac{x^\gamma}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi}a_\alpha(-\xi)y^{\alpha-\gamma}\varphi(y)dyd\xi.\end{aligned}$$

This completes the proof. □

Remark. You may have noticed that if we apply (4) to monomial $x^\alpha\xi^\beta$, we will get

$$\widehat{x^\alpha\xi^\beta}^W = \sum_{\gamma\le\beta} 2^{-|\beta|}\binom{\beta}{\gamma}P^\gamma Q^\alpha P^{\beta-\gamma}$$

which is different from the McCoy's formula on page 3. In particular, for example, we will get two different formula for $\widehat{x\xi^2}^W$:

$$\widehat{x\xi^2}^W = \frac{1}{2}(QP^2 + P^2Q) \qquad \text{and} \qquad \widehat{x\xi^2}^W = \frac{1}{4}(QP^2 + 2PQP + P^2Q).$$

There is no mistake: in view of the canonical commutative relation

$$[Q, P] = i\hbar \cdot \mathrm{Id},$$

we have

$$QP^2 + P^2Q = PQP + i\hbar P + PQP - i\hbar P = 2PQP,$$

and thus the two formulae for $\widehat{x\xi^2}^W$ coincide. We can also write down an expression for $\widehat{x\xi^2}^W$ which looks even more symmetric:

$$\widehat{x\xi^2}^W = \frac{1}{3}(QP^2 + PQP + P^2Q).$$

In what follows we will prove that such symmetric formula holds for the Weyl quantization of any monomial.

¶ Case (B) with $C = \mathrm{Id}$: Adding x to ξ.

To prove the special case of (B) where $C = \mathrm{Id}$, we need to check:

Theorem 2.4. *We have*

$$e^{-i\frac{|x|^2}{2\hbar}} \widehat{\xi^\alpha}^W e^{i\frac{|x|^2}{2\hbar}} = \widehat{(x+\xi)^\alpha}^W.$$

Proof. Let's compute

$$\begin{aligned} e^{-i\frac{|x|^2}{2\hbar}} \widehat{\xi^\alpha}^W e^{i\frac{|x|^2}{2\hbar}} \varphi &= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\frac{|x|^2}{2\hbar}} e^{i\frac{(x-y)\cdot\xi}{\hbar}} \xi^\alpha e^{i\frac{|y|^2}{2\hbar}} \varphi(y)dyd\xi \\ &= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot(\xi - \frac{x+y}{2})}{\hbar}} \xi^\alpha \varphi(y)dyd\xi \\ &= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}} \left(\xi + \frac{x+y}{2}\right)^\alpha \varphi(y)dyd\xi \\ &= \widehat{(x+\xi)^\alpha}^W \varphi. \end{aligned}$$

□

¶ Application: Symmetry in Weyl quantization.

Since we know $\widehat{\xi^\alpha} = P^\alpha$, Theorem 2.4 allows us to compute the Weyl quantization of $(x+\xi)^\alpha$ directly, and the result is neat:

Corollary 2.5. *We have*

$$\widehat{(x+\xi)^\alpha}^W = (Q+P)^\alpha.$$

Proof. We have seen that $\widehat{\xi^\alpha}^W = P^\alpha$. So we only need to check

$$e^{-i\frac{|x|^2}{2\hbar}} P^\alpha e^{i\frac{|x|^2}{2\hbar}} \varphi = (Q+P)^\alpha \varphi,$$

which follows easily from the fact

$$e^{-i\frac{|x|^2}{2\hbar}} P e^{i\frac{|x|^2}{2\hbar}} \varphi = (Q+P)\varphi$$

and the fact

$$e^{-i\frac{|x|^2}{2\hbar}} P_i P_j e^{i\frac{|x|^2}{2\hbar}} \varphi = e^{-i\frac{|x|^2}{2\hbar}} P_i e^{i\frac{|x|^2}{2\hbar}} e^{-i\frac{|x|^2}{2\hbar}} P_j e^{i\frac{|x|^2}{2\hbar}} \varphi.$$

□

Remark. Note that other t-quantizations like the Kohn-Nirenberg quantization does not satisfy this property. For example,

$$\widehat{(x+\xi)^2}^{KN} = Q^2 + 2QP + P^2 \neq (Q+P)^2.$$

More generally, one by proving case (B) for general C and then taking C to be diagonal matrices, one can easily prove: for any $a, b \in \mathbb{R}^n$,

$$\widehat{(ax+b\xi)^\alpha}^W = (aQ+bP)^\alpha,$$

where we used the abbreviation

$$(ax+b\xi)^\alpha = (a_1x_1+b_1\xi_1)^{\alpha_1} \cdots (a_nx_n+b_n\xi_n)^{\alpha_n}$$

and

$$(aQ+bP)^\alpha = (a_1Q_1+b_1P_1)^{\alpha_1} \circ \cdots \circ (a_nQ_n+b_nP_n)^{\alpha_n}.$$

The proof will be left as a simple exercise. This has a further interesting application, namely the Weyl quantization *is* the most symmetric way to quantize monomials:

Corollary 2.6.

$$\widehat{x^\alpha \xi^\beta}^W = \frac{\alpha!\beta!}{|\alpha+\beta|!} \sum_{Y_1,\cdots,Y_{|\alpha|+|\beta|}} Y_1 Y_2 \cdots Y_{|\alpha|+|\beta|},$$

where Y_1, Y_2, $\cdots Y_{|\alpha|+|\beta|}$ range over all tuples which contains α_1 copies of Q_1, α_2 copies of Q_2, $\cdots$, and β_n copies of P_n.

Proof. Comparing the coefficients of $a^\alpha b^\beta$ of both sides of

$$\widehat{(ax+b\xi)^{\alpha+\beta}}^W = (aQ+bP)^{\alpha+\beta},$$

we get

$$\widehat{x^\alpha \xi^\beta}^W = \frac{\alpha!\beta!}{(\alpha+\beta)!} \sum_{Y_1,\cdots,Y_{|\alpha|+|\beta|}} Y_1 Y_2 \cdots Y_{|\alpha|+|\beta|},$$

where Y_1, Y_2, $\cdots Y_{\alpha_1+\beta_1}$ range over all tuples which contains α_1 copies of Q_1, β_1 copies of P_1, $Y_{\alpha_1+\beta_1+1}$, $\cdots Y_{\alpha_2+\beta_2}$ range over all tuples which contains α_2 copies of Q_2, β_2 copies of P_2 and so on. The conclusion follows from the fact "P_i, Q_i commutes with P_j, Q_j for $j \neq i$" and an elementary combinatorics argument. □