LECTURE 8 — 10/19/2020
WEYL QUANTIZATION VIA LINEAR EXPONENTIALS

1. Quantizing linear exponentials

¶ From polynomials to the exponential function.

Last time, by using symplectic invariance we proved that the Weyl quantization has many nice properties on polynomials, e.g.

$$\widehat{(x+\xi)^\alpha}^W = (Q+P)^\alpha.$$

In PSet 2 you will be asked to prove similar expressions like

$$\widehat{(ax+b\xi)^\alpha}^W = (aQ+bP)^\alpha$$

and

$$\widehat{(a\cdot x+b\cdot\xi)^n}^W = (a\cdot Q+b\cdot P)^n.$$

Note that in the first expression we used abbreviations

$$(ax+b\xi)^\alpha = (a_1x_1+b_1\xi_1)^{\alpha_1}\cdots(a_nx_n+b_n\xi_n)^{\alpha_n}$$

while in the second expression,

$$(a\cdot x+b\cdot\xi)^n = (a_1x_1+\cdots+a_nx_n+b_1\xi_1+\cdots+b_n\xi_n)^n.$$

As a consequence of the second fact (together with the Taylor expansion of the exponential function), *formally* we would expect to have

$$\widehat{e^{a\cdot x+b\cdot\xi}}^W = e^{a\cdot Q+b\cdot P}.$$

Of course we need to justify the meaning of $e^{a\cdot Q+b\cdot P}$. We can't just formally define

$$e^{a\cdot Q+b\cdot P} = \sum_{k\ge 0}\frac{1}{k!}(a\cdot Q+b\cdot P)^k$$

because the operators Q and P are unbounded and we will encounter convergence problem. However, in what follows we will show that for $a,b\in\mathbb{R}^n$, we may define the operator

$$e^{it(a\cdot Q+b\cdot P)/\hbar}$$

which is a well-defined (unitary) operator. [This is a special case of *Stone's theorem* that we will discuss later.]

¶ DETOUR: The Baker-Campbell-Hausdorff formula: a special case.

To understand the operator $e^{it(a\cdot Q+b\cdot P)/\hbar}$, let's start with some general discussion.

Suppose A is a bounded linear operator defined on a Hilbert space $\mathcal{H}$. Then one can define the exponential of A to be the operator

$$e^A := \sum_{k\geq 0}\frac{1}{k!}A^k$$

which is also a bounded linear operator on $\mathcal{H}$. It is easy to check

$$\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A.$$

It follows that for any $\varphi_0 \in \mathcal{H}$, the function $\varphi(t) := e^{tA}\varphi_0$ solves the equation

$$\begin{cases} \frac{d}{dt}\varphi(t) = A\varphi(t), \\ \varphi(0) = \varphi_0. \end{cases}$$

We say the operator e^{tA} is the *solution operator* to the above equation.

Now we assume A and B are bounded linear operators. Then $A+B$ is again bounded and we have

$$e^{A+B} = \sum_{k\geq 0}\frac{1}{k!}(A+B)^k.$$

However, due to the non-commutativity of operator composition, in general

$$e^{A+B} \neq e^A e^B.$$

There is a formula called the *Baker-Campbell-Hausdorff formula*[1] which describes the relation between e^{A+B} and e^Ae^B. Here we prove a special case:

Theorem 1.1 (Baker-Campbell-Hausdorff formula, a special case)**.** *Suppose A, B are bounded linear operator on $\mathcal{H}$ and*

$$[A,[A,B]] = 0, \quad [B,[A,B]] = 0,$$

then

$$e^{A+B} = e^{-[A,B]/2}e^Ae^B. \tag{1}$$

Proof. We will prove: for any $t \in \mathbb{R}$,

$$e^{t(A+B)} = e^{-t^2[A,B]/2}e^{tA}e^{tB}.$$

We calculate the derivative of the right hand side via the Leibniz rule:

$$\frac{d}{dt}(e^{-t^2[A,B]/2}e^{tA}e^{tB}) = -t[A,B]e^{-t^2[A,B]/2}e^{tA}e^{tB} + e^{-t^2[A,B]/2}Ae^{tA}e^{tB} + e^{-t^2[A,B]/2}e^{tA}Be^{tB}.$$

Since $[A,B]$ commutes with A and thus with e^{tA}, we have

$$\frac{d}{dt}(e^{tA}Be^{-tA}) = e^{tA}(AB-BA)e^{-tA} = [A,B]$$

[1]See my Lie group notes for the general Baker-Campbell-Hausdorff formula on Lie groups.

and thus by integration,

$$e^{tA}Be^{-tA} = B + t[A,B].$$

It follows (using the fact "$[A,B]$, and thus $e^{-t^2[A,B]/2}$, commutes with everything")

$$\begin{aligned}\frac{d}{dt}(e^{-t^2[A,B]/2}e^{tA}e^{tB}) &= (-t[A,B]+A+B+t[A,B])e^{-t^2[A,B]/2}e^{tA}e^{tB}\\ &= (A+B)e^{-t^2[A,B]/2}e^{tA}e^{tB}.\end{aligned}$$

It follows that for any $\varphi_0 \in \mathcal{H}$, both $\varphi(t) = e^{t(A+B)}\varphi_0$ and $\varphi(t) = e^{-t^2[A,B]/2}e^{tA}e^{tB}\varphi_0$ solve the equation

$$\frac{d}{dt}\varphi(t) = (A+B)\varphi(t)$$

with the same initial condition $\varphi(0) = \varphi_0$. So we conclude $e^{t(A+B)} = e^{-t^2[A,B]/2}e^{tA}e^{tB}$ holds for all t and thus the theorem is proved. □

¶ The operator $e^{it(a\cdot Q+b\cdot P)/\hbar}$.

Suppose $a, b \in \mathbb{R}^n$. Before we study the operator $e^{it(a\cdot Q+b\cdot P)/\hbar}$, let's first look at two simpler operators $e^{ita\cdot Q/\hbar}$ and $e^{itb\cdot P/\hbar}$.

- We can define $e^{ita\cdot Q/\hbar}$ to be the solution operator to the equation

$$\begin{cases}\frac{\partial}{\partial t}\varphi(t,x) = \frac{ia\cdot Q}{\hbar}\varphi(t,x),\\ \varphi(0,x) = \varphi_0(x).\end{cases}$$

  [For simplicity, we always start with $\varphi_0 \in \mathscr{S}(\mathbb{R}^n)$.] It is trivial to check

$$e^{ita\cdot Q/\hbar} = \text{multiplication by the function } e^{ita\cdot x/\hbar} \tag{2}$$

  In other words, we have

$$\widehat{e^{ita\cdot x/\hbar}}^W = e^{ita\cdot Q/\hbar}. \tag{3}$$

- Similarly we define $e^{itb\cdot P/\hbar}$ to be the solution operator to the equation

$$\begin{cases}\frac{\partial}{\partial t}\varphi(t,x) = \frac{ib\cdot P}{\hbar}\varphi(t,x),\\ \varphi(0,x) = \varphi_0(x).\end{cases}$$

  It is also trivial to check that the solution is

$$(e^{itb\cdot P/\hbar}\varphi)(x) = \varphi(x+tb). \tag{4}$$

  Moreover, by the Fourier inversion formula,

$$\begin{aligned}(\widehat{e^{itb\cdot\xi/\hbar}}^W\varphi)(x) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x-y)\cdot\xi}{\hbar}}e^{i\frac{tb\cdot\xi}{\hbar}}\varphi(y)dyd\xi\\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{(x+tb-y)\cdot\xi}{\hbar}}\varphi(y)dyd\xi = \varphi(x+tb)\end{aligned}$$

  So we still have

$$\widehat{e^{itb\cdot\xi/\hbar}}^W = e^{itb\cdot P/\hbar}. \tag{5}$$

Now we are ready to study the operator $e^{it(a\cdot Q+b\cdot P)/\hbar}$, which by definition, is the solution operator to the equation

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t,x) = \frac{i(a\cdot Q+b\cdot P)}{\hbar}\varphi(t,x), \\ \varphi(0,x) = \varphi_0(x). \end{cases}$$

In general, equations of this type could be hard to solve because the non-commutativity of the operators involved. However, for the position operator Q and the momentum operator P, the canonical commutative relation gives us

$$[a\cdot Q, b\cdot P] = \sum_k a_k b_k [Q_k, P_k] = -\frac{\hbar}{i} a\cdot b \ \mathrm{Id},$$

which commutes with any operator. It turns out that the Baker-Campbell-Hausdorff formula alluded to above still holds, namely,

$$e^{\frac{it(a\cdot Q+b\cdot P)}{\hbar}} = e^{-[\frac{ita\cdot Q}{\hbar},\frac{itb\cdot P}{\hbar}]/2} e^{\frac{ita\cdot Q}{\hbar}} e^{\frac{itb\cdot P}{\hbar}}.$$

To see this, let's first compute

$$e^{-[\frac{ita\cdot Q}{\hbar},\frac{itb\cdot P}{\hbar}]/2} e^{\frac{ita\cdot Q}{\hbar}} e^{\frac{itb\cdot P}{\hbar}}\varphi(x) = e^{\frac{it^2}{2\hbar}a\cdot b} e^{\frac{it}{\hbar}a\cdot x}\varphi(x+tb).$$

Now we prove

Proposition 1.2. *We have*

$$(6) \qquad (e^{it(a\cdot Q+b\cdot P)/\hbar}\varphi)(x) = e^{\frac{it^2}{2\hbar}a\cdot b} e^{\frac{it}{\hbar}a\cdot x}\varphi(x+tb).$$

Proof. This follows from a direct computation:

$$\begin{aligned}
\frac{\partial}{\partial t}\left[e^{\frac{it^2}{2\hbar}a\cdot b} e^{\frac{it}{\hbar}a\cdot x}\varphi(x+tb)\right] =& \frac{i}{\hbar}a\cdot x\left[e^{\frac{it^2}{2\hbar}a\cdot b} e^{\frac{it}{\hbar}a\cdot x}\varphi(x+tb)\right] \\
&+ \frac{it}{\hbar}a\cdot b\left[e^{\frac{it^2}{2\hbar}a\cdot b} e^{\frac{it}{\hbar}a\cdot x}\varphi(x+tb)\right] + \left[e^{\frac{it^2}{2\hbar}a\cdot b} e^{\frac{it}{\hbar}a\cdot x} b\cdot\nabla\varphi(x+tb)\right] \\
=& \frac{i}{\hbar}a\cdot x\left[e^{\frac{it^2}{2\hbar}a\cdot b} e^{\frac{it}{\hbar}a\cdot x}\varphi(x+tb)\right] + e^{\frac{it^2}{2\hbar}a\cdot b}(b\cdot\nabla)\left[e^{\frac{it}{\hbar}a\cdot x}\varphi(x+tb)\right] \\
=& (\frac{i}{\hbar}a\cdot Q + \frac{i}{\hbar}b\cdot P)\left[e^{\frac{it^2}{2\hbar}a\cdot b} e^{\frac{it}{\hbar}a\cdot x} b\cdot\nabla\varphi(x+tb)\right].
\end{aligned}$$

□

As a consequence (and as we can expect), we also have

Corollary 1.3.

$$(7) \qquad \widehat{e^{it(a\cdot x+b\cdot\xi)/\hbar}}^{W} = e^{it(a\cdot Q+b\cdot P)/\hbar}.$$

Proof. To see this, we calculate

$$\begin{aligned}\widehat{e^{it(a\cdot x+b\cdot\xi)/\hbar}}^W\varphi(x) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi}e^{\frac{it}{\hbar}(a\cdot\frac{x+y}{2}+b\cdot\xi)}\varphi(y)dyd\xi \\ &= \frac{1}{(2\pi\hbar)^n}e^{\frac{it}{2\hbar}a\cdot x}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y+tb)\cdot\xi}e^{\frac{it}{2\hbar}a\cdot y}\varphi(y)dyd\xi \\ &= \frac{1}{(2\pi\hbar)^n}e^{\frac{it}{2\hbar}a\cdot x}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi}e^{\frac{it}{2\hbar}a\cdot(y+tb)}\varphi(y+tb)dyd\xi \\ &= e^{\frac{it^2}{2\hbar}a\cdot b}e^{\frac{it}{\hbar}a\cdot x}\varphi(x+tb).\end{aligned}$$

□

¶ Weyl quantization: Weyl's definition.

Now we are ready to give a different way to define the Weyl quantization $\widehat{a}^W$, which is in fact Weyl's original definition!

Given a symbol $a(x,\xi)$ (which could be a tempered distribution), consider the semiclassical Fourier transform $(\mathcal{F}_\hbar)_{(x,\xi)\to(y,\eta)}$,

$$[(\mathcal{F}_\hbar)_{(x,\xi)\to(y,\eta)}a](y,\eta) = \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}(x\cdot y+\xi\cdot\eta)}a(x,\xi)dxd\xi. \tag{8}$$

The Fourier inversion formula gives

$$a(x,\xi) = \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(y\cdot x+\eta\cdot\xi)}[(\mathcal{F}_\hbar)_{(x,\xi)\to(y,\eta)}a](y,\eta)dyd\eta.$$

To quantize a function f, Weyl wrote[2]:

> A quantity f is consequently carried over from classical to quantum mechanics in accordance with the rule: *replace p and q in Fourier development (14.8) of f by the Hermitian operators representing them in quantum mechanics.*

In other words, Weyl quantize the function $a(x,\xi)$ to the operator

$$\widehat{a}^W = \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(y\cdot Q+\eta\cdot P)}[(\mathcal{F}_\hbar)_{(s,\xi)\to(y,\eta)}a](y,\eta)dyd\eta, \tag{9}$$

which, when acting on a Schwartz function φ, yields

$$(\widehat{a}^W\varphi)(x) = \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}} \left[e^{\frac{i}{\hbar}(y\cdot Q+\eta\cdot P)}\varphi\right](x)[(\mathcal{F}_\hbar)_{(s,\xi)\to(y,\eta)}a](y,\eta)dyd\eta.$$

We have to show that Weyl's original definition coincides with the one we gave earlier:

[2] c.f. H. Weyl, *The Theory of Groups and Quantum Mechanics*, page 275, Dover, 1950. The first German edition was published in 1928.

Theorem 1.4. *The two definitions of Weyl quantization coincides:*

$$\widehat{a}^W = \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(y\cdot Q+\eta\cdot P)}[(\mathcal{F}_\hbar)_{(s,\xi)\to(y,\eta)}a](y,\eta)dyd\eta.$$

Proof of Theorem 1.4. Applying the right hand side of (9) to φ, and using Proposition 1.2, we get

$$\begin{aligned}
&\frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} \left[e^{\frac{i}{\hbar}(y\cdot Q+\eta\cdot P)}\varphi\right](x)[(\mathcal{F}_\hbar)_{(s,\xi)\to(y,\eta)}a](y,\eta)dyd\eta \\
=&\frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}y\cdot x+\frac{i}{2\hbar}y\cdot\eta}\varphi(x+\eta)e^{-\frac{i}{\hbar}(s\cdot y+\xi\cdot\eta)}a(s,\xi)dsd\xi dyd\eta \\
=&\frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}y\cdot(\frac{x}{2}+\frac{\tau}{2}-s)}e^{\frac{i}{\hbar}\xi\cdot(x-\tau)}\varphi(\tau)a(s,\xi)dsd\xi dyd\tau \qquad (\eta\to\tau=x+\eta) \\
=&\frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}y\cdot\zeta}e^{\frac{i}{\hbar}\xi\cdot(x-\tau)}\varphi(\tau)a(\frac{x+\tau}{2}-\zeta,\xi)d\zeta dyd\tau d\xi \qquad (s\to\zeta=\frac{x+\tau}{2}-s) \\
=&\frac{1}{(2\pi\hbar)^{n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\xi\cdot(x-\tau)}\varphi(\tau)a(\frac{x+\tau}{2},\xi)d\tau d\xi,
\end{aligned}$$

where in the last step we used the Fourier inversion formula. □

2. DETOUR: Some Functional Analysis

In this section we list some definitions and theorems related to self-adjoint operators from functional analysis. For details, c.f. *Reed-Simon, Methods of modern mathematical physics Vol I*, or *B. Hall, Quantum Theory for Mathematicians*.

¶ Self-adjointness.

Definition 2.1. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{D}(A)\subset\mathcal{H}$ a subspace.

(1) A linear operator $A:\mathcal{D}(A)\to\mathcal{H}$ is called *closed* if its graph
$$\operatorname{graph}(A)=\{(x,Ax)\mid x\in\mathcal{D}(A)\}$$
is a closed subspace of $\mathcal{H}\times\mathcal{H}$.
(2) We say $B:\mathcal{D}(B)\to\mathcal{H}$ an *extension* of A if $\mathcal{D}(A)\subset\mathcal{D}(B)$ and $Ax=Bx$ for all $x\in\mathcal{D}(A)$. In this case we will denote $A\subset B$.
(3) For any linear operator $A:\mathcal{D}(A)\to\mathcal{H}$, the smallest closed extension of A (if exists) is called the *closure* of A and is denoted by $\overline{A}$.
(4) The *adjoint* of a densely defined linear operator $A:\mathcal{D}(A)\to\mathcal{H}$ is an linear operator $A^*:\mathcal{D}(A^*)\to\mathcal{H}$ so that
$$\langle Ax,y\rangle=\langle x,A^*y\rangle,$$
where $\mathcal{D}(A^*):=\{y\in\mathcal{H}\mid\exists C>0 \text{ s.t. } \langle Ax,y\rangle\le C\|x\| \text{ for } \forall x\in\mathcal{D}(A)\}$.
(5) A densely defined linear operator $A:\mathcal{D}(A)\to\mathcal{H}$ is *symmetric* if $A\subset A^*$.
(6) A densely defined linear operator $A:\mathcal{D}(A)\to\mathcal{H}$ is *self-adjoint* if $A=A^*$.

(7) A densely defined linear operator $A : \mathcal{D}(A) \to \mathcal{H}$ is *essentially self-adjoint* if $\overline{A} = A^*$.

A useful criteria for a symmetric operator to be self-adjoint is

Theorem 2.2. *Let A be a symmetric operator on $\mathcal{H}$. Then*

(1) A is essentially self-adjoint if and only if the images of the operators $A \pm i$ are dense in $\mathcal{H}$.
(2) A is self-adjoint if and only if the images of the operators $A \pm i$ are $\mathcal{H}$.

One of the most important theorems for self-adjoint operators acting on a Hilbert space $\mathcal{H}$ is

Theorem 2.3 (The spectral theorem (multiplication form))**.** *Suppose A is a self-adjoint operator on $\mathcal{H}$. Then there is measurable space (X, μ), a measurable real-valued function h on X, and a unitary map $V : \mathcal{H} \to L^2(X, \mu)$ such that A is unitary equivalent via V to the operator "multiplication by $h(x)$" on $L^2(X, \mu)$, namely*

$$V \circ A \circ V^*(\psi)(x) = h(x)\psi(x). \tag{10}$$

As a consequence, for any measurable function f on $\mathbb{R}$ (or on $\sigma(A)$), we may define an operator $f(A)$ via

$$f(A) = V^* \circ M_{f(h(x))} \circ V.$$

In general this might be complicated and might have a very small domain if f is unbounded. However, if f is a bounded function, then $f(A)$ will be a bounded linear operator on $\mathcal{H}$. In particular, if we take $f(x) = e^{itx}$, then the operator

$$e^{itA} = V^* \circ M_{e^{ith(x)}} \circ V \tag{11}$$

is defined on the whole of $\mathcal{H}$ and is unitary.

As a consequence of the spectral theorem (multiplication form), together with PSet 1-8 we immediately get the following theorem which will be used later in the proof of Weyl law:

Corollary 2.4 (Helffer-Sjöstrand)**.** *Let P be a self-adjoint operator on $\mathcal{H}$, f be a Schwartz function and $\tilde{f}$ be an almost analytic extension of f. Then*

$$f(P) = \frac{1}{\pi i} \int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z)(P - z)^{-1} dm. \tag{12}$$

¶ Stone's theorem.

We need to justify that the operators e^{itA} defined by the spectral theorem coincides with the ones defined as the solution operator to suitable partial differential equations. This is known as the Stone's theorem.

Definition 2.5. Let $\mathcal{H}$ be a Hilbert space.

(1) A *one-parameter unitary group* $U(t)$, $t \in \mathbb{R}$, is a family of unitary operators on $\mathcal{H}$ such that $U(0) = \mathrm{Id}$ and
$$U(s+t) = U(s)U(t), \quad \forall s, t \in \mathbb{R}.$$
(2) A one-parameter unitary group $U(t)$ is said to be (*strongly continuous* if
$$\lim_{t \to t_0} U(t)x = U(t_0)x$$
for each $t_0 \in \mathbb{R}$ and each $x \in \mathcal{H}$.

For example, one can check that the three families
$$e^{ita \cdot Q/\hbar}, \quad e^{itb \cdot P/\hbar}, \quad e^{it(a \cdot Q + b \cdot P)/\hbar}$$
form three strongly continuous one-parameter unitary groups.

It turns out that self-adjoint operators on $\mathcal{H}$ and strongly continuous one-parameter unitary groups on $\mathcal{H}$ are closely related to each other:

Theorem 2.6 (Stone's Theorem)**.** *Suppose $A : \mathcal{D}(A) \to \mathcal{H}$ is self-adjoint. Then*
$$U(t) = e^{itA}$$
is a strongly continuous one-parameter unitary group on $\mathcal{H}$, and satisfies
$$\lim_{h \to 0} \frac{1}{i} \frac{U(t+\hbar)\psi - U(t)\psi}{h} = AU(t)\psi = U(t)A\psi \tag{13}$$
for all $\psi \in \mathrm{Dom}(A)$ and all $t \in \mathbb{R}$.

Conversely, if $U(t)$ is a strongly continuous one-parameter unitary group on $\mathcal{H}$, then the operator A (called the infinitesimal generator *of $U(t)$) defined via*
$$A\psi = \lim_{h \to 0} \frac{1}{i} \frac{U(h)\psi - \psi}{h}$$
is densely defined and self-adjoint, and $U(t) = e^{itA}$.

This justifies our definitions of $e^{ita \cdot Q/\hbar}, e^{itb \cdot P/\hbar}$ and $e^{it(a \cdot Q + b \cdot P)/\hbar}$ as the solution operators to suitable partial differential equations.

¶ A functional calculus.

We just mentioned how to define $f(A)$ via the spectral theorem. There is however another way to define $f(A)$ via the unitary group $U(t) = e^{itA}$, which is very similar to the way Weyl define his quantization: start with the Fourier inversion formula
$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itx} \hat{f}(t) dt.$$
and define
$$f(A) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{itA} \hat{f}(t) dt. \tag{14}$$
Of course in this definition one need to assume f to be a function whose Fourier transform $\hat{f} = \mathcal{F} f \in L^1$.