LECTURE 9 — 10/21/2020
THE COMPOSITION FORMULA

1. The composition formula for Weyl quantization

¶ The composition.

Given two semiclassical pseudo-differential operators $\widehat{a}^W$ and $\widehat{b}^W$, a natural question is: what is the composition $\widehat{a}^W \circ \widehat{b}^W$? is it still a semiclassical pseudo-differential operator? If yes, what is its Weyl symbol? Our first task today will be: given $a(x,\xi)$ and $b(x,\xi)$, find a new symbol function $a \star b = a \star b(x,\xi)$, called the *Moyal product* of a and b, so that

$$\widehat{a \star b}^W = \widehat{a}^W \circ \widehat{b}^W.$$

Let's start with a very simple example: we take $n = 1$, $a(x,\xi) = x$ and $b(x,\xi) = \xi$. Then

$$\widehat{a}^W \circ \widehat{b}^W = QP = \frac{1}{2}(PQ + QP) + \frac{1}{2}\hbar i \cdot \mathrm{Id} = \widehat{x\xi + \frac{1}{2}\hbar i}^W.$$

From this simple example we can observe

- even in this simple case the function $a \star b$ is $\hbar$-dependent:
$$a \star b(x,\xi) = x\xi + \frac{1}{2}\hbar i;$$
- the function $a \star b(x,\xi)$ we are looking for is not the product $x\xi = a(x,\xi)b(x,\xi)$ (as we can expect, otherwise the composition will be commutative);
- however, the product $x\xi$ does appear in $a \star b(x,\xi)$ as the "leading term" (namely the term with lowest $\hbar$ power).

We will show below that given a and b, we can construct a function c (which may depends on $\hbar$, with leading term is ab) such that $\widehat{a \star b}^W = \widehat{a}^W \circ \widehat{b}^W$.

For simplicity we will assume $a, b \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$. [We will extend the results to larger symbol classes later.] Last time we showed

$$\widehat{a}^W = \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(u\cdot Q + \mu \cdot P)} [\mathcal{F}_\hbar a](u,\mu)\, du d\mu$$

and similarly

$$\widehat{b}^W = \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(v\cdot Q + \nu \cdot P)} [\mathcal{F}_\hbar b](v,\nu)\, dv d\nu.$$

It follows that the composition $\widehat{a}^W \circ \widehat{b}^W$ equals

$$\frac{1}{(2\pi\hbar)^{4n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(u\cdot Q+\mu\cdot P)} e^{\frac{i}{\hbar}(v\cdot Q+\nu\cdot P)} [\mathcal{F}_\hbar a](u,\mu)[\mathcal{F}_\hbar b](v,\nu) dv d\nu du d\mu.$$

On the other hand, from the Baker-Campbell-Hausdorff formula we can guess

$$e^{\frac{i}{\hbar}(u\cdot Q+\mu\cdot P)} e^{\frac{i}{\hbar}(v\cdot Q+\nu\cdot P)} = e^{\frac{i}{2\hbar}(\mu\cdot v-u\cdot\nu)} e^{\frac{i}{\hbar}((u+v)\cdot Q+(\mu+\nu)\cdot P)}$$

which can be justified rigorously via the formula

$$(e^{it(a\cdot Q+b\cdot P)/\hbar}\varphi)(x) = e^{\frac{it^2}{2\hbar}a\cdot b} e^{\frac{it}{\hbar}a\cdot x}\varphi(x+tb).$$

It follows by changing of variables $(v,\nu) \to (z = u+v, \zeta = \mu+\nu)$ that

$$\widehat{a}^W \circ \widehat{b}^W = \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} c(z,\zeta) e^{\frac{i}{\hbar}(z\cdot Q+\zeta\cdot P)} dz d\zeta,$$

where

$$c(z,\zeta) = \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} [\mathcal{F}_\hbar a](u,\mu)[\mathcal{F}_\hbar b](z-u,\zeta-\mu) e^{\frac{i}{2\hbar}[\mu\cdot(z-u)-u\cdot(\zeta-\mu)]} du d\mu.$$

So we get

$$\begin{aligned} a\star b(x,\xi) &= (\mathcal{F}_\hbar^{-1} c)(x,\xi) \\ &= \frac{1}{(2\pi\hbar)^{4n}} \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} [\mathcal{F}_\hbar a](u,\mu)[\mathcal{F}_\hbar b](z-u,\zeta-\mu) e^{\frac{i}{2\hbar}[\mu\cdot(z-u)-u\cdot(\zeta-\mu)]} e^{\frac{i}{\hbar}(x\cdot z+\xi\cdot\zeta)} dz d\zeta du d\mu, \end{aligned}$$

and by changing variable back from (z,ζ) to (v,ν), we arrived at an "ugly" formula for the Moyal product:

(1)

$$a\star b(x,\xi) = \frac{1}{(2\pi\hbar)^{4n}} \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} [\mathcal{F}_\hbar a](u,\mu)[\mathcal{F}_\hbar b](v,\nu) e^{\frac{i}{2\hbar}(\mu\cdot v-u\cdot\nu)} e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+x\cdot v+\xi\cdot\nu)} dv d\nu du d\mu.$$

¶ A quadratic exponential.

To get a better expression for $a\star b$, we observe that by Fourier inversion formula,

$$\frac{1}{(2\pi\hbar)^{4n}} \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} [\mathcal{F}_\hbar a](u,\mu)[\mathcal{F}_\hbar b](v,\nu) e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+y\cdot v+\eta\cdot\nu)} dv d\nu du d\mu = a(x,\xi)b(y,\eta).$$

So we need a clever way to relate the product $e^{\frac{i}{2\hbar}(\mu\cdot v-u\cdot\nu)} e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+x\cdot v+\xi\cdot\nu)}$ to the function $e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+y\cdot v+\eta\cdot\nu)}$. It is not too hard to do so, at least formally: If we write

$$e^{\frac{i\hbar}{2}(D_\xi\cdot D_y - D_x\cdot D_\eta)} \text{ “}=\text{” } \sum_{k\geq 0} \frac{1}{k!}\left(\frac{i\hbar}{2}(D_\xi\cdot D_y - D_x\cdot D_\eta)\right)^k, \tag{2}$$

then

$$\begin{aligned} e^{\frac{i\hbar}{2}(D_\xi D_y - D_x D_\eta)} e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+y\cdot v+\eta\cdot\nu)} &= \sum_{k\geq 0}\frac{1}{k!}\left(\frac{i\hbar}{2}(D_\xi\cdot D_y - D_x\cdot D_\eta)\right)^k e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+y\cdot v+\eta\cdot\nu)} \\ &= \sum_{k\geq 0}\frac{1}{k!}\left(\frac{i}{2\hbar}(\mu\cdot v - u\cdot\nu)\right)^k e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+y\cdot v+\eta\cdot\nu)} \\ &= e^{\frac{i}{2\hbar}(\mu\cdot v-u\cdot\nu)} e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+y\cdot v+\eta\cdot\nu)}. \end{aligned}$$

So we get

$$\left[e^{\frac{i\hbar}{2}(D_\xi\cdot D_y - D_x\cdot D_\eta)} e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+y\cdot v+\eta\cdot\nu)}\right]_{y=x,\eta=\xi} = e^{\frac{i}{2\hbar}(\mu\cdot v-u\cdot\nu)} e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+x\cdot v+\xi\cdot\nu)}. \tag{3}$$

Let's justify that the formal computation is rigorous. Of course we need to justify the operator $e^{\frac{i\hbar}{2}(D_\xi\cdot D_y - D_x\cdot D_\eta)}$: We have a $4n$-dimensional Euclidian space $\mathbb{R}^{4n}$ with coordinates x, ξ, y, η, and we have a densely defined differential operator

$$D_\xi\cdot D_y - D_x\cdot D_\eta = \sum_{j=1}^n (D_{\xi_j}D_{y_j} - D_{x_j}D_{\eta_j}).$$

Now the operator $e^{\frac{i\hbar}{2}(D_\xi\cdot D_y - D_x\cdot D_\eta)}$ can be defined via functional calculus, as we explained last time. Equivalently, by Stone's theorem, the (unitary) operator $e^{\frac{it\hbar}{2}(D_\xi\cdot D_y - D_x\cdot D_\eta)}$ can be characterized as the solution operator to the partial differential equation

$$\partial_t(\varphi(t)) = \frac{i\hbar}{2}(D_\xi\cdot D_y - D_x\cdot D_\eta)\varphi(t). \tag{4}$$

To justify the validity of (3), we need an explicit formula for the operator $e^{\frac{i\hbar}{2}(D_\xi\cdot D_y - D_x\cdot D_\eta)}$. It turns out, as one can expect, that it is the Weyl quantization of a quadratic exponential. To write down such a quadratic exponential, we have to consider the $8n$-dimensional space $T^*\mathbb{R}^{4n} = \mathbb{R}^{4n}\times\mathbb{R}^{4n}$, whose coordinates are denoted by $x, \xi, y, \eta, \alpha, \beta, \gamma, \delta$. Now consider the following quadratic exponential:

$$\sigma_t(x, y, \xi, \eta; \alpha, \beta, \gamma, \delta) = e^{\frac{it}{2\hbar}(\gamma\cdot\beta - \alpha\cdot\delta)}.$$

(We will denote $\sigma_1 = \sigma$.) We first prove:

Lemma 1.1. $\widehat{\sigma_t}^W = e^{\frac{it\hbar}{2}(D_\xi\cdot D_y - D_x\cdot D_\eta)}$.

Proof. It is enough to check that $\widehat{\sigma_t}^W$ is the solution operator to the equation (4). This follows from direct computation: since σ_t is a function that depends only on "cotangent variables", we have (see Lecture 7)

$$\widehat{\sigma_t}^W = \mathcal{F}_\hbar^{-1}\circ\sigma_t\circ\mathcal{F}_\hbar$$

and thus

$$\begin{aligned}\partial_t(\widehat{\sigma_t}^W\varphi) &= \mathcal{F}_\hbar^{-1}\circ\partial_t\sigma_t\circ\mathcal{F}_\hbar\varphi\\ &= \mathcal{F}_\hbar^{-1}\circ\left(\frac{i}{2\hbar}(\gamma\cdot\beta-\alpha\cdot\delta)\right)\sigma_t\circ\mathcal{F}_\hbar\varphi\\ &= \mathcal{F}_\hbar^{-1}\circ\left(\frac{i}{2\hbar}(\gamma\cdot\beta-\alpha\cdot\delta)\right)\circ\mathcal{F}_\hbar\circ\mathcal{F}_\hbar^{-1}\circ\sigma_t\circ\mathcal{F}_\hbar\varphi\\ &= \frac{i\hbar}{2}(D_\xi\cdot D_y-D_x\cdot D_\eta)\circ\widehat{\sigma_t}^W\varphi.\end{aligned}$$

□

As a consequence, we get

$$\begin{aligned}&e^{\frac{it\hbar}{2}(D_\xi\cdot D_y-D_x\cdot D_\eta)}e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+y\cdot v+\eta\cdot\nu)}\\ =&\frac{1}{(2\pi\hbar)^{4n}}\int_{\mathbb{R}^{4n}}\int_{\mathbb{R}^{4n}}e^{\frac{i}{\hbar}(x-\tilde{x},y-\tilde{y},\xi-\tilde{\xi},\eta-\tilde{\eta})\cdot(\alpha,\beta,\gamma,\delta)}e^{\frac{i}{\hbar}(\tilde{x},\tilde{\xi},\tilde{y},\tilde{\eta})\cdot(u,\mu,v,\nu)}e^{\frac{i}{2\hbar}(\gamma\cdot\beta-\alpha\cdot\delta)}d\alpha d\beta d\gamma d\delta d\tilde{x}d\tilde{\xi}d\tilde{y}d\tilde{\eta}\\ =&\frac{1}{(2\pi\hbar)^{4n}}\int_{\mathbb{R}^{4n}}\int_{\mathbb{R}^{4n}}e^{\frac{i}{\hbar}(\tilde{x},\tilde{y},\tilde{\xi},\tilde{\eta})\cdot(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta})}e^{\frac{i}{\hbar}(x,y,\xi,\eta)\cdot(u-\tilde{\alpha},\mu-\tilde{\beta},v-\tilde{\gamma},\nu-\tilde{\delta})}e^{\frac{i}{2\hbar}[(v-\tilde{\gamma})\cdot(\mu-\tilde{\beta})-(u-\tilde{\alpha})\cdot(\nu-\tilde{\delta})]}d\tilde{\alpha}d\tilde{\beta}d\tilde{\gamma}d\tilde{\delta}d\tilde{x}d\tilde{\xi}d\tilde{y}d\tilde{\eta}\\ =&e^{\frac{i}{2\hbar}(\mu\cdot v-u\cdot\nu)}e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+y\cdot v+\eta\cdot\nu)}.\end{aligned}$$

So the formula (3) is justified, and thus the "ugly formula" (1) can be simplified to

$$\begin{aligned}a\star b(x,\xi) &= \frac{1}{(2\pi\hbar)^{4n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}[\mathcal{F}_\hbar a](u,\mu)[\mathcal{F}_\hbar b](v,\nu)\left[e^{\frac{i\hbar}{2}(D_\xi\cdot D_y-D_x\cdot D_\eta)}e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+y\cdot v+\eta\cdot\nu)}\right]_{y=x,\eta=\xi}dvd\nu dud\mu.\\ &= e^{\frac{i\hbar}{2}(D_\xi\cdot D_y-D_x\cdot D_\eta)}\left[\frac{1}{(2\pi\hbar)^{4n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}[\mathcal{F}_\hbar a](u,\mu)[\mathcal{F}_\hbar b](v,\nu)e^{\frac{i}{\hbar}(x\cdot u+\xi\cdot\mu+y\cdot v+\eta\cdot\nu)}dvd\nu dud\mu\right]_{y=x,\eta=\xi}\\ &= \left[e^{\frac{i\hbar}{2}(D_\xi\cdot D_y-D_x\cdot D_\eta)}(a(x,\xi)b(y,\eta))\right]_{y=x,\eta=\xi}.\end{aligned}$$

We conclude

Theorem 1.2. *Suppose $a,b\in\mathscr{S}(\mathbb{R}^n\times\mathbb{R}^n)$, then*

$$\widehat{a}^W\circ\widehat{b}^W(x,\hbar D)=\widehat{a\star b}^W, \tag{5}$$

where the Moyal product $a\star b$ *is given by*

$$a\star b(x,\xi)=\left[e^{\frac{i\hbar}{2}(D_\xi\cdot D_y-D_x\cdot D_\eta)}(a(x,\xi)b(y,\eta))\right]\Big|_{y=x,\eta=\xi}. \tag{6}$$

¶ Asymptotic behavior of $a\star b$.

As we explained at the beginning of today's lecture, the Moyal product $a\star b$ is a function that depends on $\hbar$. Fortunately, the $\hbar$-dependence is not too bad: at least formally, by applying (2) we will get

$$a\star b(x,\xi)\sim\sum_{k=0}^{\infty}\frac{1}{k!}\frac{(i\hbar)^k}{2^k}\left(D_\xi\cdot D_y-D_x\cdot D_\eta\right)^k[a(x,\xi)b(y,\eta)]\Big|_{y=x,\eta=\xi},$$

in other words, $a\star b$ is a nice asymptotic series in $\hbar$. To justify this formula, we need to apply the stationary phase expansion that we studied in Lecture 5.

First according to Proposition 1.1 in Lecture 7, for any non-singular symmetric matrix Q,

$$(\widehat{e^{\frac{i}{2\hbar}\xi^T Q\xi}}^W\varphi)(x) = \frac{|\det Q|^{-1/2}}{(2\pi\hbar)^{n/2}} e^{i\frac{\pi}{4}\mathrm{sgn}Q}\int_{\mathbb{R}^n} e^{-\frac{i}{2\hbar}y^T Q^{-1}y}\varphi(x+y)dy. \tag{7}$$

In particular, if we raise n to $4n$, take

$$Q = \frac{1}{2}\begin{pmatrix} & & & -I \\ & & I & \\ & I & & \\ -I & & & \end{pmatrix} \quad\rightsquigarrow\quad Q^{-1} = 2\begin{pmatrix} & & & -I \\ & & I & \\ & I & & \\ -I & & & \end{pmatrix}, \mathrm{sgn}(Q)=0$$

replace x by x,ξ,y,η, and replace ξ by $\alpha,\beta,\gamma,\delta$, we get

$$e^{\frac{i\hbar}{2}(D_\xi\cdot D_y - D_x\cdot D_\eta)}\varphi(x,\xi,y,\eta) = \frac{1}{(\pi\hbar)^{2n}}\int_{\mathbb{R}^{4n}} e^{-\frac{2i}{\hbar}(\tilde\xi\cdot\tilde y-\tilde x\cdot\tilde\eta)}\varphi(x+\tilde x,\xi+\tilde\xi,y+\tilde y,\eta+\tilde\eta)d\tilde x d\tilde\xi d\tilde y d\tilde\eta$$

and thus

$$a\star b(x,\xi) = \left[\frac{1}{(\pi\hbar)^{2n}}\int_{\mathbb{R}^{4n}} e^{-\frac{2i}{\hbar}(\tilde\xi\cdot\tilde y-\tilde x\cdot\tilde\eta)}a(x+\tilde x,\xi+\tilde\xi)b(y+\tilde y,\eta+\tilde\eta)d\tilde x d\tilde\xi d\tilde y d\tilde\eta\right]_{y=x,\eta=\xi}.$$

Now we apply the exact stationary phase formula for oscillating integrals with quadratic phase (Theorem 1.3 in Lecture 5), namely

$$\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}x^TQx}a(x)dx \sim (2\pi\hbar)^{n/2}\frac{e^{i\frac{\pi}{4}\mathrm{sgn}(Q)}}{|\det Q|^{1/2}}\sum_{k=0}^\infty \frac{1}{k!}\left(-\frac{i\hbar}{2}p_{Q^{-1}}(D)\right)^k a(0).$$

to our setting (with n raised to $4n$, Q replaced by $-Q^{-1}$ etc) to get

$$a\star b(x,\xi) \sim \left[\sum_{k\geq 0}\frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \left(D_{\tilde\xi}\cdot D_{\tilde y} - D_{\tilde x}\cdot D_{\tilde\eta}\right)^k\Big|_{\tilde x=0,\cdots,\tilde\eta=0}\left[a(x+\tilde x,\xi+\tilde\xi)b(y+\tilde y,\eta+\tilde\eta)\right]\right]_{y=x,\eta=\xi}$$

and thus we conclude

Theorem 1.3. *Let $a,b\in\mathscr{S}$. Then as $\hbar\to 0$,*

$$a\star b(x,\xi) \sim \sum_{k=0}^\infty \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k (D_\xi\cdot D_y - D_x\cdot D_\eta)^k\left[a(x,\xi)b(y,\eta)\right]\Big|_{y=x,\eta=\xi}. \tag{8}$$

An immediate consequence of (8) is

Corollary 1.4. *If $supp(a)\cap supp(b)=\varnothing$, then $a\star b = O(\hbar^\infty)$.*

Proof. If $supp(a)\cap supp(b)=\varnothing$, then each term in the expansion (8) vanishes. □

Remark. If a,b are polynomials in ξ-variables (namely, if the operators $\widehat{a}^W$ and $\widehat{b}^W$ are both semiclassical differential operators), then the asymptotic formula formula (8) will be a polynomial expansion in $\hbar$ and thus be an exact formula.

2. Quantization condition

¶ Asymptotic quantization condition in general.

Another application of (8) is to show that the Weyl quantization satisfies the quantization condition we mentioned in Lecture 3, in the semi-classical sense:

Theorem 2.1. *The Weyl quantization satisfies the quantization condition in the semiclassical sense:*

$$\big[\widehat{a}^W, \widehat{b}^W\big] = \frac{\hbar}{i}\widehat{\{a,b\}}^W + O(\hbar^3), \tag{9}$$

where $\{a,b\}$ is the Poisson bracket of a and b that we defined in Lecture 2.

Proof. We have

$$\begin{aligned} a\star b &= ab + \frac{i\hbar}{2}(D_\xi a\cdot D_y b - D_x a\cdot D_\eta b)|_{y=x,\eta=\xi} + O(\hbar^2)\\ &= ab + \frac{\hbar}{2i}(\partial_\xi a\cdot \partial_x b - \partial_x a\cdot\partial_\xi b) + O(\hbar^2)\\ &= ab + \frac{\hbar}{2i}\{a,b\} + O(\hbar^2). \end{aligned}$$

It follows

$$[\widehat{a}^W, \widehat{b}^W] = \widehat{a}^W\circ\widehat{b}^W - \widehat{b}^W\circ\widehat{a}^W = \widehat{(a\star b - b\star a)}^W = \frac{\hbar}{i}\widehat{\{a,b\}}^W + O(\hbar^2).$$

Finally notice that the $\hbar^2$-term of $a\star b$,

$$\frac{1}{2}\frac{(i\hbar)^2}{2^2}(D_\xi\cdot D_y - D_x\cdot D_\eta)^2[a(x,\xi)b(y,\eta)]|_{y=x,\eta=\xi},$$

equals the $\hbar^2$-term in $b\star a$, since

$$(D_\xi\cdot D_y - D_x\cdot D_\eta)^2 = (D_x\cdot D_\eta - D_\xi\cdot D_y)^2.$$

So the $O(\hbar^2)$ can be improved to $O(\hbar^3)$. □

¶ Quantization condition for linear symbol.

Consider a special case: $a(x,\xi) = c\cdot x + d\cdot\xi$ is a function that is linear in both x and ξ. Then for any $b\in\mathscr{S}$, if we formally apply (8) we will see that the only nonvanishing terms in the asymptotic expansion are the terms with $k=0$ and $k=1$, and we will get

$$\big[\widehat{a}^W, \widehat{b}^W\big] = \frac{\hbar}{i}\widehat{\{a,b\}}^W + O(\hbar^\infty).$$

It turns out that the *exact* quantization rule holds in this case (without the $O(\hbar^\infty)$-term):

Proposition 2.2. *Suppose $a(x,\xi) = c\cdot x + d\cdot\xi$. Then for any $b\in\mathscr{S}$,*

$$\left[\widehat{a}^W, \widehat{b}^W\right] = \frac{\hbar}{i}\widehat{\{a,b\}}^W.$$

We will leave the rigourous proof as an exercise. Note that it is enough to prove the following two special cases:

$$[Q_j, \widehat{b}^W] = -\hbar\widehat{D_{\xi_j}b}^W \qquad \text{and} \qquad [P_j, \widehat{b}^W] = \hbar\widehat{D_{x_j}b}^W.$$

¶ Quantization condition for polynomials of degree ≤ 2.

For Weyl quantization we can do a little bit further. In Theorem 2.1 we showed that the difference

$$\left[\widehat{a}^W, \widehat{b}^W\right] - \frac{\hbar}{i}\widehat{\{a,b\}}^W = O(\hbar^3)$$

since the $\hbar^2$ cancelled. As a consequence, if $a(x,\xi)$ is a quadratic polynomial in x,ξ (namely $a(x,\xi)$ is a linear combination of $x_j, \xi_j, x_jx_k, x_j\xi_k$ and $\xi_j\xi_k$), then the asymptotic expansion (8) should also terminate after the first two terms with $k=0$ and $k=1$. As a result, we would have a further *exact* quantization rule:

Proposition 2.3. *Suppose $a(x,\xi)$ is a quadratic polynomial in x,ξ. Then $\forall b\in\mathscr{S}$,*

$$\left[\widehat{a}^W, \widehat{b}^W\right] = \frac{\hbar}{i}\widehat{\{a,b\}}^W.$$

Again a rigourous proof will be left as an exercise.

Remark. In Lecture 3 (and PSet 1-3) we have already seen that the same proposition fails for cubic polynomials.

3. Other quantizations

¶ Other quantizations.

Finally we list analogous properties for other t-quantizations. The start point is the following observation: the t-quantization of linear exponentials can be easily computed via the definition (repeating the proof of Corollary 1.3 in Lecture 8):

$$\widehat{e^{\frac{i}{\hbar}(a\cdot x+b\cdot\xi)}}^t(\varphi)(x) = e^{\frac{i}{\hbar}(1-t)a\cdot b}e^{\frac{i}{\hbar}a\cdot x}\varphi(x+b) \tag{10}$$

This fact has two consequences:

- First, as in the Weyl case, we have an alternative formula to define $\widehat{a}^t$ through the t-quantization of linear exponentials:

$$\widehat{a}^t = \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}} \widehat{e^{\frac{i}{\hbar}(y\cdot x+\eta\cdot\xi)}}^t(\mathcal{F}_\hbar a)(y,\eta)dyd\eta. \tag{11}$$

For a proof, one only need to repeat the proof of Theorem 1.4 in Lecture 8. [1]

- There is a very simple way to change quantization for linear exponentials:

$$\widehat{e^{\frac{i}{\hbar}(a\cdot x+b\cdot\xi)}}^{t} = e^{\frac{i}{\hbar}(s-t)a\cdot b}\ \widehat{e^{\frac{i}{\hbar}(a\cdot x+b\cdot\xi)}}^{s}. \tag{12}$$

In particular, we have

$$\widehat{e^{\frac{i}{\hbar}(a\cdot x+b\cdot\xi)}}^{t} = e^{\frac{i}{\hbar}(\frac{1}{2}-t)a\cdot b}\ e^{\frac{i}{\hbar}(a\cdot Q+b\cdot P)}.$$

As a consequence of (11) and (12), we get the following "change of quantization formula" which tells us that different t-quantizations gives us the same set of operators[2], and moreover, it tells us how to "jump" from one t-quantization to another:

Theorem 3.1 (Change of quantizations)**.** *For any $0 \le s, t \le 1$ and any $a \in \mathscr{S}$, if we let*

$$b(x,\xi) = e^{i(t-s)\hbar D_x\cdot D_\xi} a(x,\xi), \tag{13}$$

then

$$\widehat{b}^t = \widehat{a}^s.$$

Proof. First we explain the meaning of the operator $e^{i(t-s)\hbar D_x\cdot D_\xi}$: it can be defined via the functional calculus, or equivalently, as the solution operator to a corresponding differential equation. Moreover, if we repeat the same argument on page 3, we immediately see that $e^{i(t-s)\hbar D_x\cdot D_\xi}$ is the Weyl quantization of the $4n$-variable function

$$\sigma(x,\xi;y,\eta) = e^{\frac{i}{\hbar}(t-s)y\cdot\eta}.$$

Now the conclusion follows:

$$\begin{aligned} Op_\hbar^t(b) &= \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}} \widehat{e^{\frac{i}{\hbar}(y\cdot x+\eta\cdot\xi)}}^{t} (\mathcal{F}_\hbar e^{i(t-s)\hbar D_x\cdot D_\xi} a)(y,\eta)dyd\eta \\ &= \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}} \widehat{e^{\frac{i}{\hbar}(y\cdot x+\eta\cdot\xi)}}^{s} e^{\frac{i}{\hbar}(s-t)y\cdot\eta}(\mathcal{F}_\hbar \mathcal{F}_\hbar^{-1} e^{\frac{i}{\hbar}(t-s)y\cdot\eta}\mathcal{F}_\hbar a)(y,\eta)dyd\eta \\ &= Op_\hbar^s(a). \end{aligned}$$

□

[1] Note that this formula is essentially a consequence of linearity of quantization maps: We wrote a as a "superposition" of linear exponentials via the Fourier transform, and then $\widehat{a}^t$ is the "superposition" of the quantization of corresponding linear exponentials!

[2] This is why these operators are all called "semiclassical pseudodifferential operators" without specifying which t we are using. However, I should warn you that still it is possible that even if a is independent of $\hbar$, the resulting symbol b after changing quantization may depend on $\hbar$.

¶ The product formula and quantization rule for t-quantizations.

In general, by a similar computation one can write down the composition formula for t-quantizations:

$$\widehat{a}^t \circ \widehat{b}^t = \widehat{a \star_t b}^t,$$

where the *t-Moyal product* $a \star_t b$ is the function

(14)

$$\begin{aligned}(a \star_t b)(x,\xi) &= e^{i\hbar(D_\xi \cdot D_v - D_u D_\eta)}(a(tx+(1-t)u,\xi)b((1-t)y+tv,\eta))|_{y=u=v=x,\eta=\xi} \\ &\sim \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{k!}(D_\xi \cdot D_v - D_u D_\eta)^k (a(tx+(1-t)u,\xi)b((1-t)y+tv,\eta))|_{y=u=v=x,\eta=\xi}.\end{aligned}$$

Although the general formula is complicated, we can still easily see

$$\begin{aligned}a \star_t b - b \star_t a &= \frac{\hbar}{i}(t\partial_\xi a \cdot \partial_x b - (1-t)\partial_x a \cdot \partial_\xi b) - \frac{\hbar}{i}(t\partial_\xi b \cdot \partial_x a - (1-t)\partial_x b \cdot \partial_\xi a) + O(\hbar^2) \\ &= \frac{\hbar}{i}\{a,b\} + O(\hbar^2)\end{aligned}$$

and thus

$$[\widehat{a}^t, \widehat{b}^t] = \frac{\hbar}{i}\widehat{\{a,b\}}^t + O(\hbar^2).$$

Note that in this case the error is $O(\hbar^2)$, not as good as the Weyl quantization.

¶ The product formula for the standard KN-quantizations.

In particular, by taking $t=1$, we get a much simpler composition formula:

Theorem 3.2. *Suppose $a, b \in \mathscr{S}$, then*

$$\widehat{a}^{KN} \circ \widehat{b}^{KN} = \widehat{(a \star_{KN} b)}^{KN}, \tag{15}$$

where $a \star_{KN} b$ is the function

$$\begin{aligned}(a \star_{KN} b)(x,\xi) &= e^{i\hbar D_\xi \cdot D_y}(a(x,\xi)b(y,\eta))|_{y=x,\eta=\xi} \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}\tilde{x}\cdot\tilde{\xi}} a(x,\xi+\tilde{\xi})b(x+\tilde{x},\xi)d\tilde{x}d\tilde{\xi} \\ &\sim \sum_{k=0}^{\infty}\frac{(i\hbar)^k}{k!}(D_\xi \cdot D_y)^k(a(x,\xi)b(y,\eta))|_{y=x,\eta=\xi}.\end{aligned} \tag{16}$$