LECTURE 10 — 10/26/2020
QUANTIZING GENERAL SYMBOLS

1. Semi-classical symbols

Today we would like to extend the theory of semiclassical pseudo-differential operators with Schwartz functions as symbols to more general symbols which may depends on $\hbar$ and more importantly, which may grow as $x, \xi \to \infty$. The key observation is the following: According to the fact

$$
\begin{aligned}
e^{\frac{i}{\hbar}(x-y)\cdot\xi} &= (1+|\xi|^2)^{-1}[1+(\hbar D_y)^2](e^{\frac{i}{\hbar}(x-y)\cdot\xi}) \\
&= (1+|x-y|^2)^{-1}[1+(\hbar D_\xi)^2](e^{\frac{i}{\hbar}(x-y)\cdot\xi}),
\end{aligned} \tag{1}
$$

we may gain factors $\approx |\xi|^{-m}$ or $\approx |x-y|^{-m}$ after (formally) applying integration by parts arguments many times. As a result, we may allow our symbol $a(x,\xi)$ to grow in x or ξ, as long as the growth rate is *under control*, namely, in an "at most polynomially" way.

¶ Order functions.

Now we introduce a conception that can be used to describe such a polynomial growth. For simplicity we will use the notation [1]

$$\langle z\rangle := (1+|z|^2)^{1/2}$$

for $z \in \mathbb{R}^d$.

Definition 1.1. A measurable function $g : \mathbb{R}^d \to \mathbb{R}^+$ is called an *order function* if there exists constants C, N so that for any $z, w \in \mathbb{R}^d$,

$$m(w) \le C\langle z-w\rangle^N m(z). \tag{2}$$

Note that if we take $z = 0$, we immediately get

$$m(w) \le C\langle w\rangle^N$$

for some constant C and N. So order functions grow at most polynomially.

From the definition one immediately get

Lemma 1.2. *We have*

(1) If m_1, m_2 are order functions, so are the functions $m_1 + m_2$ and $m_1 m_2$.
(2) If m is an order function, so is the function m^a for any $a \in \mathbb{R}$.

[1] Sometimes it is called the "Japanese bracket" of z. It behave like $|z|$ for large z, but it has the advantage that it is smooth and non-vanishing as $z \to 0$.

Some simple examples of order functions:

- $m(z) = 1$.
- $m(z) = \langle z \rangle$.
- $m(z) = \langle z' \rangle$, where $z' = (z_1, \cdots, z_l)$ for some $l \le d$.
- In particular, if $d = 2n$ and $a, b \in \mathbb{R}$, then the functions
$$m(x,\xi) = \langle x \rangle^a \langle \xi \rangle^b \qquad \text{and} \qquad \tilde{m}(x,\xi) = \langle x \rangle^a + \langle \xi \rangle^b$$
are order functions.

Remark. In microlocal analysis, usually we take $d = 2n$ and thus $\mathbb{R}^d_z = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, in which case the most widely used order function is $\langle \xi \rangle^N$, which, as we will see later, has the advantage that the corresponding symbol class (see definition below) is invariant under coordinate changes and thus can be defined on manifolds. However, we do use other order functions in semiclassical analysis.

¶ The need for $\hbar$-dependent symbols.

We have seen last time that the Moyal product of two $\hbar$-independent symbols will be $\hbar$-dependent in general. Here is another example showing the necessity for introducing $\hbar$-dependent symbols:

Example. The usual (non-semiclassical) differential operator
$$P = \sum_{|\alpha| \le m} a_\alpha D^\alpha$$
can be written as
$$P = \hbar^{-m} \sum_{j=0}^{m} \hbar^j \sum_{|\alpha| = m-j} a_\alpha (\hbar D)^\alpha,$$
and thus is a semiclassical pseudo-differential operator with Kohn-Nirenberg symbol
$$p(x,\xi,\hbar) = \hbar^{-m} \sum_{j=0}^{m} \hbar^j \sum_{|\alpha| = m-j} a_\alpha(x) \xi^\alpha.$$
(If we don't allow $\hbar$-dependence, we will not have these operators in our class.)

¶ Symbol classes.

Now we are ready to define symbol classes associated to an order function. In what follows we will assume $\hbar$-dependence for the symbol function a.

Definition 1.3. Let m be an order function on $\mathbb{R}^d$.

(1) We say $a \in C^\infty(\mathbb{R}^d)$ is in the *symbol class* $S(m)$ associated to m if for any multi-index $\alpha \in \mathbb{N}^d$, there exists constant $C_\alpha > 0$ such that for all $z \in \mathbb{R}^d$,
$$|\partial^\alpha a(z)| \le C_\alpha m(z). \tag{3}$$
(2) We say $a = a(z, \hbar)$ is in the symbol class $S(m)$ if there exists $\hbar_0 > 0$ so that for $0 < \hbar < \hbar_0$, the function $a = a(\cdot, \hbar)$ is in $S(m)$ and the constant C_α in (3) is uniform for all $0 < \hbar < \hbar_0$.

(3) For any $k \in \mathbb{R}$ and $0 \le \delta \le 1$ we let $S^k_\delta(m)$ be the space of functions $a(z, \hbar)$ which belongs to $S(m)$ for each $\hbar$ and satisfies

$$|\partial^\alpha a(z, \hbar)| \le C_\alpha \hbar^{-\delta|\alpha|-k} m(z), \tag{4}$$

where the constants C_α is again uniform in $\hbar$.

For example, the class $S(1)$ contains all uniformly bounded smooth functions with all derivatives uniformly bounded over $\mathbb{R}^{2n}$ (and with respect to $\hbar \in (0, \hbar_0]$).

We will abbreviate $S_\delta(m) = S^0_\delta(m)$, $S = S(1)$, and $S_\delta = S_\delta(1)$,

Remark. Obviously $\mathscr{S} \subset S(m)$ holds for any order function m. Moreover,

(1) As in the case of Schwartz functions, one could define semi-norms on $S(m)$ to make it into a Fréchet space. Under this topology, the multiplication map

$$\bullet : S(m_1) \times S(m_2) \to S(m_1 m_2), \qquad (a_1, a_2) \to a_1 a_2$$

is continuous. Similar result holds for the spaces $S^k_\delta(m)$.

(2) Moreover, the inclusion $\mathscr{S} \subset S(m)$ is dense in the topology of $S(\langle z\rangle^\varepsilon m)$ for any $\varepsilon > 0$. To see this, for each $a \in S(m)$ one just take

$$a_j(z) = \chi(\frac{z}{j}) a(z)$$

for some $\chi \in \mathscr{S}$ with $\chi(0) = 1$.

Note that in the case $\delta = 0$, we have $S_0(m) = S(m)$. We will omit the subscript δ in $S^k_\delta(m)$ if $\delta = 0$. So for each $k \in \mathbb{R}$ we have

$$S^{-k}(m) = \hbar^k S(m).$$

Obviously if $k_1 < k_2$, then $S^{-k_1}(m) \supset S^{-k_2}(m)$. In what follows we will denote

$$S^{-\infty}(m) = \bigcap_k S^{-k}(m).$$

Note that a symbol $a \in S^{-\infty}(m)$ if and only if for any multi-index $\alpha \in \mathbb{N}^d$ and any $N \in \mathbb{N}$, there exists constant $C_{\alpha,N}$ so that

$$|\partial^\alpha a(z, \hbar)| \le c_{\alpha,N} \hbar^N m(z).$$

In particular, $a \in S^k_\delta$ for any δ and any k. It follows that for any δ,

$$S^{-\infty}(m) = \bigcap_k S^{-k}_\delta(m). \tag{5}$$

2. Asymptotic analysis II

¶ Asymptotic series for general symbols.

Now we extend the conception of asymptotic series that we introduced at the beginning of Lecture 5 to asymptotic series of functions that lie in a "decreasing sequence of symbol classes associated to an order function m". Let

$$k_0 < k_1 < k_2 < \cdots < k_j < \cdots \to \infty$$

be a sequence of increasing real numbers that tends to ∞.

Definition 2.1. If $a_j \in S_\delta^{-k_j}(m)$, we say $a \in S_\delta^{-k_0}(m)$ is *asymptotic to* $\sum_{j=0}^\infty a_j$, and write

$$a \sim \sum_{j=0}^\infty a_j,$$

if for every $N \in \mathbb{N}$,

$$a - \sum_{j=0}^N a_j \in S_\delta^{-k_{N+1}}(m). \tag{6}$$

Note that the similar conceptions introduced in Lecture 5 is a special case with $m = 1$ and $k_j = j$. Again we don't require the formal series $\sum a_j$ to be convergent for any $\hbar$.

Remark. Obviously $a \sim 0$ if and only if $a \in S^{-\infty}(m)$.

¶ Borel's Lemma.

The following theorem, named after E. Borel, is crucial in asymptotic analysis, which implies that every power series is the Taylor series of some smooth function:

Theorem 2.2 (Borel Lemma)**.** *For any sequence $a_j \in S_\delta^{-k_j}(m)$, where k_j is a strictly increasing sequence that tends to infinity, there exists a symbol $a \in S_\delta^{-k_0}(m)$ so that*

$$a \sim \sum_{j=0}^\infty a_j.$$

Moreover, a is unique up to an element in $S^{-\infty}(m)$.

Proof. Uniqueness is obvious.

To prove the existence, we choose a smooth cut-off function χ so that

- $0 \le \chi \le 1$ on $\mathbb{R}$,
- $\chi \equiv 1$ on $[0, 1]$,
- $\chi \equiv 0$ on $[2, \infty)$.

Define

$$a(x) = \sum_j \chi(\lambda_j \hbar) a_j(x), \tag{7}$$

where λ_j is a sequence to be determined below. We will choose λ_j so that $\lambda_j \to \infty$ as $j \to \infty$, with some additional restrictions. Note the fact $\lambda_j \to \infty$ implies that for each $\hbar$ the sum (7) is a finite sum. In particular, a is smooth for each $\hbar$.

Additional restriction on λ_j: For each multi-index α with $|\alpha| \le j$, we have

$$\chi(\lambda_j\hbar)|\partial^\alpha a_j| \le C_{j,\alpha}\hbar^{k_j-\delta|\alpha|}\chi(\lambda_j\hbar)m \le 2^{k_j-k_{j-1}}C_{j,\alpha}\frac{\hbar^{k_{j-1}-\delta|\alpha|}}{\lambda_j^{k_j-k_{j-1}}}m$$

where we used the fact $\chi(\lambda_j\hbar)(\lambda_j\hbar)^{k_j-k_{j-1}} \le 2^{k_j-k_{j-1}}$. We will choose λ_j large enough so that $\lambda_j > \lambda_{j-1}$ and so that for all $|\alpha| \le j$,

$$\frac{2^{k_j-k_{j-1}}C_{j,\alpha}}{\lambda_j^{k_j-k_{j-1}}} \le 2^{-j}.$$

It follows that for $j \ge |\alpha|$,

$$\chi(\lambda_j\hbar)|\partial^\alpha a_j| \le \hbar^{k_{j-1}-\delta|\alpha|}2^{-j}m. \tag{8}$$

It remains to prove $a \in S_\delta^{-k_0}(m)$ and $a \sim \sum a_j$. Fix any α, we have

$$\begin{aligned}|\partial^\alpha a| &\le \sum_{j\le|\alpha|}|\partial^\alpha a_j| + \sum_{j>|\alpha|}|\chi(\lambda_j\hbar)\partial^\alpha a_j| \\ &\le \sum_{j\le|\alpha|}C_{j,\alpha}\hbar^{k_j-\delta|\alpha|}m + \sum_{j>|\alpha|}\hbar^{k_{j-1}-\delta|\alpha|}2^{-j}m \\ &\le C_\alpha\hbar^{k_0-\delta|\alpha|}m,\end{aligned}$$

where we took $C_\alpha = \sum_{j\le|\alpha|}C_{j,\alpha}+1$. This proves $a \in S_\delta^{-k_0}(m)$.

To prove $a \sim \sum a_j$, we do similar calculations: For any α and any N, we have

$$\begin{aligned}\left|\partial^\alpha(a - \sum_{j=0}^{N-1}a_j)\right| &\le \sum_{j=N+|\alpha|+1}^{\infty}|\partial^\alpha a_j|\chi(\lambda_j\hbar) + \sum_{j=N}^{N+|\alpha|}|\partial^\alpha a_j| + \sum_{j=0}^{N-1}|\partial^\alpha a_j|(1-\chi(\lambda_j\hbar)) \\ &\le \sum_{j=N+|\alpha|+1}^{\infty}\hbar^{k_{j-1}-\delta|\alpha|}2^{-j}m + \sum_{j=N}^{N+|\alpha|}C_{j,\alpha}\hbar^{k_j-\delta|\alpha|}m + \sum_{j=0}^{N-1}C_{j,\alpha}\hbar^{k_j-\delta|\alpha|}(1-\chi(\lambda_j\hbar))m \\ &\le \hbar^{k_N-\delta|\alpha|}m + \sum_{j=N}^{N+|\alpha|}C_{j,\alpha}\hbar^{k_N-\delta|\alpha|}m + \sum_{j=0}^{N-1}C_{j,\alpha}\hbar^{k_j-\delta|\alpha|}(\lambda_N\hbar)^{k_N-k_j}m \\ &\le C_{\alpha,N}\hbar^{k_N-\delta|\alpha|}m\end{aligned}$$

for the constant

$$C_{\alpha,N} = 1 + \sum_{j=N}^{N+|\alpha|}C_{j,\alpha} + \sum_{j=0}^{N-1}C_{j,\alpha}\lambda_N^{k_N-k_j},$$

where in the third line we used the fact that for $0 \le j \le N-1$:

- if $\hbar\lambda_N \le 1$, then $1-\chi(\lambda_j\hbar) = 0$;
- if $\hbar\lambda_N \ge 1$, then $1-\chi(\lambda_j\hbar) \le 1 \le (\hbar\lambda_N)^{k_N-k_j}$.

This completes the proof. □

Remark. In many applications one take $k_j = j$, in which case the theorem says that for any sequence $a_j \in S_\delta(m)$, up to $S^{-\infty}(m)$ there exists a unique asymptotic sum

$$a \sim \sum_{j=0}^{\infty} \hbar^j a_j$$

in $S_\delta(m)$.

¶ An Application: WKB solution.

Consider the 1-dimensional semiclassical Schrödinger equation

$$(-\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V(x))u = Eu.$$

In Lecture 3 we have seen how to solve this equation for $V(x) = x^2/2$, in which case a non-trivial solution exists if and only if $E = (2n+1)\hbar$, and the solutions u are closely related to the Hermite polynomials.

In general, one can't hope to find such an exact solution. However, it is still possible to find an approximate solution, the so-called WKB solution[2], of the form

$$u(x) \sim e^{-i\phi(x)/\hbar} \sum_{k\geq 0} \hbar^k a_k(x)$$

with $a_0 \neq 0$, which solve the equation asymptotically:

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V(x) - E\right) u \sim 0.$$

For simplicity we will only find an approximate solution inside the region

$$\{x : V(x) < E\},$$

which is usually called the *classical allowed region*). So we suppose ϕ is a function defined on an interval I inside the classical allowed region. To find the function ϕ and a_k's, we first compute

$$-\frac{\hbar^2}{2}\frac{d^2}{dx^2}(e^{-i\phi(x)/\hbar}a) = e^{-i\phi/\hbar}\left[\frac{\hbar}{2}i\phi'' a + \frac{1}{2}(\phi')^2 a + i\hbar\phi' a' - \frac{\hbar^2}{2}a''\right].$$

[2]WKB is the abbreviation of three mathematicians: Wentzel, Kramers and Brillouin.

So after pluging in the asymptotic expansion of u into the equation, we get

$$
\begin{aligned}
&(-\frac{\hbar^2}{2}\frac{d^2}{dx^2}+V(x)-E)\left(e^{-i\phi(x)/\hbar}\sum_{k\geq 0}\hbar^k a_k(x)\right)\\
=\ & e^{-i\phi(x)/\hbar}\times\Big[\frac{1}{2}(\phi')^2 a_0+V(x)a_0-Ea_0\\
&\qquad +\hbar(\frac{i}{2}\phi'' a_0+i\phi' a_0'+(\frac{1}{2}(\phi')^2+V-E)a_1)\\
&\qquad +\sum_{k\geq 2}\hbar^k(\frac{i}{2}\phi'' a_{k-1}+i\phi' a_{k-1}'-\frac{1}{2}a_{k-2}''+(\frac{1}{2}(\phi')^2+V-E)a_k)\Big].
\end{aligned}
$$

Since $a_0 \neq 0$, the leading term gives us (known as the *eikonal equation*)

$$\frac{1}{2}(\phi')^2+V(x)-E=0. \tag{9}$$

As a consequence, we get two different solutions of ϕ:

$$\phi_\pm(x)=\pm\int\sqrt{E-V(x)}dx.$$

Note that the eikonal equation also simplifies the remaining equations (known as the *transport equation* that determines a_k's:

$$\frac{i}{2}\phi'' a_0+i\phi' a_0'=0 \tag{10}$$

which determines a_0 (which depends on the choice of $\phi_\pm$), and

$$\frac{i}{2}\phi'' a_{k-1}+i\phi' a_{k-1}'-\frac{1}{2}a_{k-2}''=0 \tag{11}$$

for $k \geq 2$, which determines the remaining a_k's (again depends on the choice of $\phi_\pm$). Finally as a consequence of the Borel theorem, we get two asymptotic solutions:

$$u_\pm(x)\sim e^{-i\phi_\pm(x)/\hbar}\sum_{k\geq 0}\hbar^k a_k^\pm(x).$$

Remark. For the *classical forbidden region* $\{x \ : V(x) > E\}$, one can use similar idea to find an asymptotic solution of the form

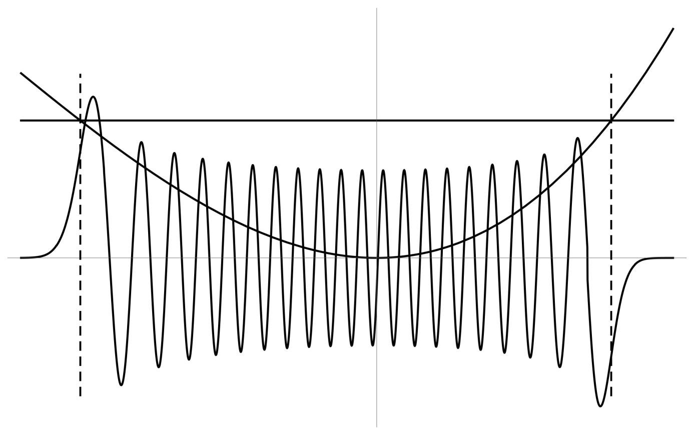
$$e^{-\phi(x)/\hbar}\sum_j \hbar^j a_j(x)$$

which solve the equation asymptotically (in a stronger sense):

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dx^2}+V(x)-E\right)u=O(\hbar^\infty)e^{-\phi(x)/\hbar}.$$

The solution near the *turning points* $V(x_0)=E$ is more complicated. (Ref: Dimassi and Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit*).

Here is a picture from Wikipedia showing the oscillation of the solution inside the classical allowed region and its "exponential decaying" inside the classical forbidden region.



3. Quantization of general symbols

¶ Quantizing general symbols.

Now let $a \in S^k_\delta(\mathbb{R}^{2n})$. We quantize a to get a semiclassical pseudo-differential operator as before. For example, the Kohn-Nirenberg (standard) quantization of a is

$$[\widehat{a}^{KN}\varphi](x) := \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi}a(x,\xi)\varphi(y)dyd\xi$$

while the Weyl quantization of a is

$$[\widehat{a}^{W}\varphi](x) := \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi}a(\frac{x+y}{2},\xi)\varphi(y)dyd\xi.$$

We have seen that if $a \in \mathscr{S}$ is a Schwartz function, then the semiclassical t-quantization $Op^t_\hbar(a)$ maps $\mathscr{S}'$ continuously into $\mathscr{S}$, and if $a \in \mathscr{S}'$ is a tempered distribution, then the operator $Op^t_\hbar(a)$ maps $\mathscr{S}$ continuously into $\mathscr{S}'$. As a result, for very general tempered distributional symbols, one can't composite semiclassical pseudodifferential operators.

Now we prove that For a symbol $a \in S^k_\delta$, the corresponding semiclassical pseudodifferential operators map $\mathscr{S}$ into $\mathscr{S}$ and map $\mathscr{S}'$ into $\mathscr{S}'$. We will handle the L^2 theory later. For simplicity we take $k = 0$ and use the Weyl quantization here.

Theorem 3.1. *If $a \in S_\delta(m)$, then*

$$\widehat{a}^W : \mathscr{S} \to \mathscr{S}$$

and

$$\widehat{a}^W : \mathscr{S}' \to \mathscr{S}'.$$

Moreover, both maps are continuous linear maps.

Proof. Let N be a constant so that $m(z) \le C\langle z\rangle^N$.

$\boxed{\text{Step 1:}}$ $\widehat{a}^W$ maps $\mathscr{S}$ into L^∞.
According to the facts

$$\begin{aligned} e^{\frac{i}{\hbar}(x-y)\cdot\xi} &= (1+|\xi|^2)^{-1}[1+(\hbar D_y)^2](e^{\frac{i}{\hbar}(x-y)\cdot\xi}) \\ &= (1+|x-y|^2)^{-1}[1+(\hbar D_\xi)^2](e^{\frac{i}{\hbar}(x-y)\cdot\xi}), \end{aligned}$$

we get, for $\varphi \in \mathscr{S}$,

$$\begin{aligned} [\widehat{a}^W\varphi](x) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(1+|\xi|^2)^{-N-n}[1+(\hbar D_y)^2]^{N+n}(e^{\frac{i}{\hbar}(x-y)\cdot\xi})a(\frac{x+y}{2},\xi)\varphi(y)dyd\xi \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{\frac{i}{\hbar}(x-y)\cdot\xi}(1+|\xi|^2)^{-N-n}[1+(\hbar D_y)^2]^{N+n}\left[a(\frac{x+y}{2},\xi)\varphi(y)\right]dyd\xi \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{\frac{i}{\hbar}(x-y)\cdot\xi}(1+|x-y|^2)^{-N}\times \\ &\qquad [1+(\hbar D_\xi)^2]^N\left\{(1+|\xi|^2)^{-N-n}[1+(\hbar D_y)^2]^{N+n}\left[a(\frac{x+y}{2},\xi)\varphi(y)\right]\right\}dyd\xi. \end{aligned}$$

Since $a \in S_\delta(m)$, there exists constants C and M so that for any multi-indices α, β with $|\alpha| \le 2N$ and $|\beta| \le 2N+2n$,

$$|D_\xi^\alpha D_y^\beta a(\frac{x+y}{2},\xi)| \le \hbar^M \widetilde{C} m(\frac{x+y}{2},\xi) \le C\hbar^M\langle\frac{x+y}{2}\rangle^N\langle\xi\rangle^N.$$

Also since $\varphi \in \mathscr{S}$, there exists constant C so that for all $|\alpha| \le 2N+2n$,

$$|D^\alpha\varphi(y)| \le \frac{C}{(1+4|y|^2)^{N+n}}.$$

Using the inequality

$$(1+|x-y|^2)(1+4|y|^2) \ge 1+\left(\frac{|x+y|}{2}\right)^2$$

one can see that there exists constant C so that

$$|[\widehat{a}^W\varphi](x)| \le C\left(\int_{\mathbb{R}^n}(1+|\xi|^2)^{-n}d\xi\right)\left(\int_{\mathbb{R}^n}(1+4|y|^2)^{-n}dy\right) \le C'.$$

In other words, $\widehat{a}^W$ maps $\mathscr{S}$ into L^∞.

$\boxed{\text{Step 2:}}$ $x^\alpha\widehat{a}^W$ maps $\mathscr{S}$ into L^∞.
Since

$$[x^\alpha\widehat{a}^W\varphi](x) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\hbar D_\xi + y)^\alpha e^{\frac{i}{\hbar}(x-y)\cdot\xi}a(\frac{x+y}{2},\xi)u(y)dyd\xi,$$

the same integration by parts argument proves that $x^\alpha\widehat{a}^W$ maps $\mathscr{S}$ into L^∞.

$\boxed{\text{Step 3:}}$ $\partial^\beta \circ \widehat{a}^W$ maps $\mathscr{S}$ into L^∞.
Recall that if $b(\xi,-x) = a(x,\xi)$, then

$$\widehat{a}^W = (\mathcal{F}_\hbar)^{-1}_{\xi\to x}\circ\widehat{b}^W\circ(\mathcal{F}_\hbar)_{x\to\xi}.$$

It follows that for any β,

$$(\hbar D)^\beta \widehat{a}^W = (\mathcal{F}_\hbar)^{-1}_{\xi\to x} \circ \xi^\beta \widehat{b}^W \circ (\mathcal{F}_\hbar)_{x\to\xi}.$$

But

$$\begin{aligned}\varphi \in \mathscr{S} &\Longrightarrow (\mathcal{F}_\hbar)_{x\to\xi}\varphi \in \mathscr{S}\\ &\Longrightarrow \langle\xi\rangle^{n+1}\xi^\beta \widehat{b}^W \circ (\mathcal{F}_\hbar)_{x\to\xi}\varphi \in L^\infty\\ &\Longrightarrow \xi^\beta \widehat{b}^W (\mathcal{F}_\hbar)_{x\to\xi}\varphi \in L^1\\ &\Longrightarrow (\mathcal{F}_\hbar)^{-1}_{\xi\to x}\xi^\beta \widehat{b}^W (\mathcal{F}_\hbar)_{x\to\xi}\varphi \in L^\infty.\end{aligned}$$

So $\partial^\beta \circ \widehat{a}^W$ maps $\mathscr{S}$ into L^∞.

Step 4: $\widehat{a}^W$ maps $\mathscr{S}$ continuously to $\mathscr{S}$.

Applying the same integration by parts arguments, one can prove that $x^\alpha\partial^\beta\widehat{a}^W$ maps $\mathscr{S}$ into L^∞. This proves $\widehat{a}^W$ maps $\mathscr{S}$ to $\mathscr{S}$.

The continuity can be proved by a similar argument: if all semi-norms of $\varphi_j\mathscr{S}$ tends to 0 as $j\to\infty$, so do the semi-norms of $\widehat{a}^W u_j$.

Step 6: $\widehat{a}^W$ maps $\mathscr{S}'$ to $\mathscr{S}'$.

Finally we use the fact that for any $u, v \in \mathscr{S}$,

$$\langle \widehat{a}^W u, v\rangle = \langle u, \widehat{\bar{a}}^W v\rangle.$$

Since $\bar{a} \in S_\delta(m)$, for any $v \in \mathscr{S}$ we have $\widehat{\bar{a}}^W v \in \mathscr{S}$. So the above formula tells us that $\widehat{a}^W u$ is a well-defined tempered distribution for $u \in \mathscr{S}'$. The continuity of the map $\widehat{a}^W : \mathscr{S}' \to \mathscr{S}'$ follows from the continuity of the map $\widehat{a}^W : \mathscr{S} \to \mathscr{S}$. □

¶ The composition law.

Last time we proved that if $a, b \in \mathscr{S}$, then

$$\widehat{a}^W \circ \widehat{b}^W = \widehat{a\star b}^W,$$

where

$$a\star b(x,\xi) = e^{\frac{i\hbar}{2}(D_\xi\cdot D_y - D_x\cdot D_\eta)}(a(x,\xi)b(y,\eta))\Big|_{y=x,\eta=\xi}. \tag{12}$$

By using the same technique as in the proof of Theorem 3.1, one can check that the same formula holds for symbols in $S_\delta(m)$[3]: namely if $a \in S_\delta(m_1)$ and $b \in S_\delta(m_2)$, then

$$\widehat{a}^W \circ \widehat{b}^W = \widehat{a\star b}^W,$$

where $a\star b$ is given by the formula (12).

However, we still need to justify the asymptotic expansion

$$a\star b \sim \sum_{k=0}^\infty \frac{1}{k!}\frac{(i\hbar)^k}{2^k}\left(D_\xi\cdot D_y - D_x\cdot D_\eta\right)^k\left[a(x,\xi)b(y,\eta)\right]\Big|_{y=x,\eta=\xi}.$$

[3]Another way to see this: since $\mathscr{S}$ is dense in $S_\delta(m)$, the result should follows from the continuity of quantization.

For this purpose we need to extend properties of the operator $\widehat{e^{\frac{i}{2\hbar}\xi^T Q\xi}}^W$ to general symbols:

Theorem 3.2. *Let Q be a symmetric non-singular $2n\times 2n$ real matrix, let $0 \le \delta < \frac{1}{2}$, and let m be an order function. Then the operator $\widehat{e^{\frac{i}{2\hbar}\xi^T Q\xi}}^W$ maps $S_\delta(m)$ to $S_\delta(m)$. Moreover, for every $a \in S_\delta(m)$ one has*

$$\widehat{e^{\frac{i}{2\hbar}\xi^T Q\xi}}^W a \sim \sum_{k=0}^\infty \frac{1}{k!}\frac{(i\hbar)^k}{2^k} p_Q(D)^k a. \tag{13}$$

Proof. We choose a partition of unity $\{\chi_k\}$ subordinate to the covering

$$\{B(0,k+1)\setminus B(0,k-1) \mid k = 1,2,\cdots\}$$

of $\mathbb{R}^{2n}$. According to Proposition 1.1 in Lecture 7, there exists a constant C so that

$$\begin{aligned}\widehat{e^{\frac{i}{2\hbar}\xi^T Q\xi}}^W a(z) &= \frac{C}{\hbar^n}\int_{\mathbb{R}^{2n}} e^{-\frac{i}{2\hbar}p_{Q^{-1}(w)}} a(z+w)dw \\ &= \sum_{k=1}^\infty \frac{C}{\hbar^n}\int_{\mathbb{R}^{2n}} e^{-\frac{i}{2\hbar}p_{Q^{-1}(w)}}\chi_k(w) a(z+w)dw.\end{aligned}$$

Since each $\chi_k(w)a(z+w)$ is compactly supported in w, and the only critical point of $p_{Q^{-1}(w)}$ is 0 which is outside the support of $\chi_k(w)z(z+w)$ for all $k \ge 2$, the exact stationary phase formula for quadratic phase gives (just as last time, the constant C will cancel out)

$$\frac{C}{\hbar^n}\int_{\mathbb{R}^{2n}} e^{-\frac{i}{2\hbar}p_{Q^{-1}(w)}}\chi_1(w)a(z+w)dw \sim \sum_{k=0}^\infty \frac{1}{k!}\frac{(i\hbar)^k}{2^k}p_Q(D)^k a$$

and for all $k \ge 2$,

$$\frac{C}{\hbar^n}\int_{\mathbb{R}^{2n}} e^{-\frac{i}{2\hbar}p_{Q^{-1}(w)}}\chi_k(w)a(z+w)dw = O(\hbar^\infty).$$

Since $\frac{(i\hbar)^k}{2^k}p_Q(D)^k a \in S_\delta^{-k+2k|\delta|}$, the conclusion follows from Borel lemma. □

So by repeating the arguments in last lecture we conclude

Theorem 3.3. *If $0 \le \delta < \frac{1}{2}$, $a \in S_\delta(m_1), b \in S_\delta(m_2)$, then $a \star b \in S_\delta(m_1 m_2)$.*

Proof. Clearly $c(z,w) = a(z)b(w) \in S_\delta(m_1(z)m_2(w))$. So by Theorem 3.2,

$$e^{\frac{i\hbar}{2}(D_\xi\ \cdot D_y - D_x\cdot D_\eta)}c(x,\xi,y,\eta) \in S_\delta(m_1(z)m_2(w)).$$

The conclusion follows. □

As an consequence,

Corollary 3.4. *Suppose* $0 \leq \delta < \frac{1}{2}$, *then for* $a \in S_\delta(m_1)$ *and* $b \in S_\delta(m_2)$,

$$[\widehat{a}^W, \widehat{b}^W] = \frac{\hbar}{i}\widehat{\{a,b\}}^W + O(\hbar^{3(1-2\delta)}).$$

We remark that the leading order of $\hbar$ in the function

$$\frac{\hbar}{i}\{a,b\} = \frac{\hbar}{i}\sum(\partial_{\xi_j}a\partial_{x_j}b - \partial_{\xi_j}b\partial_{x_j}a)$$

is $\hbar^{1-2\delta}$ instead of $\hbar$, since after each derivative we will lost $\hbar^\delta$. This explains why we need to assume $0 \leq \delta < \frac{1}{2}$.