LECTURE 11: $L^2$-THEORY OF SEMICLASSICAL PsDOs: BOUNDEDNESS

In the previous several lectures, we have studied the definition and basic properties of semiclassical pseudodifferential operators, but mainly as an operator acting on $\mathcal{S}(\mathbb{R}^n)$. However, as we have seen, in quantum part (=the spectral part) of the story the natural space should be a Hilbert space: $\mathcal{S}(\mathbb{R}^n)$ is not. In the next several lectures we shall study properties of semiclassical pseudodifferential operators as linear operators acting on $L^2(\mathbb{R}^n)$, or in cases we need more regularity, acting on the Sobolev spaces $H^s(\mathbb{R}^n)$.

1. $L^2$-boundedness of $\text{Op}_a(t)$ for Schwartz symbols

Suppose $a = a(x, \xi) \in \mathcal{S}(\mathbb{R}^{2n})$ is a Schwartz function. Then as we have seen, the operator $\hat{a}^W$, or more generally, the operator $\text{Op}_a(t)$ for any $t \in [0,1]$, maps $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. In particular, these operators are linear maps from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. It turns out that for a Schwartz symbol, the operator $\text{Op}_a(t)$ is always a bounded linear operator (and as we will prove next time, is a compact operator) on $L^2(\mathbb{R}^n)$. In what follows we will provide two different proofs of this fact.

\[ \text{Schur's test.} \]

To prove the $L^2$-boundedness of linear operators like $\text{Op}_a(t)$ which are defined by Schwartz kernels, a very useful criterion is the following Schur’s test:

**Lemma 1.1 (Schur’s Test).** Let $K: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a continuous function satisfying

\[
C_1 = \sup_x \int_{\mathbb{R}^n} |K(x,y)|\,dy < +\infty \quad \text{and} \quad C_2 = \sup_y \int_{\mathbb{R}^n} |K(x,y)|\,dx < +\infty,
\]

and let $A$ be the linear operator with Schwartz kernel $K$:

\[
Au(x) = \int_{\mathbb{R}^n} K(x,y)u(y)\,dy.
\]

Then $A$ is a bounded linear operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with

\[
\|A\|_{L^2(\mathbb{R}^n)} \leq (C_1C_2)^{\frac{1}{2}}
\]

**Proof.** For any $u \in L^2(\mathbb{R}^n)$ the Cauchy-Schwartz inequality gives

\[
|Au(x)|^2 \leq \int_{\mathbb{R}^n} |K(x,y)|\,dy \cdot \int_{\mathbb{R}^n} |K(x,y)||u(y)|^2\,dy \leq C_1 \int_{\mathbb{R}^n} |K(x,y)||u(y)|^2\,dy.
\]
Integrating with respect to $x$, we get
\[ \|Au\|_{L^2}^2 \leq \int_{\mathbb{R}^n} \left( C_1 \cdot \int_{\mathbb{R}^n} |K(x, y)||u(y)|^2 dy \right) dx \leq C_1 C_2 \|u\|_{L^2}^2. \]

\[ \square \]

\[ L^2 \)-boundedness for PsDOs with Schwartz symbols.\]

As an immediate consequence,

**Theorem 1.2.** If $a = a(x, \xi)$ is a Schwartz function, then for any $t \in [0, 1]$,  
\[ \text{Op}_t^a : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \]
is a bounded linear operator with  
\[ \|\text{Op}_t^a\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \sup_x \sup_{|\alpha| \leq n+1} \|\partial_\xi^\alpha a(x, \xi)\|_{L^1(\mathbb{R}_\xi^n)}. \]

**Proof.** Recall that the Schwartz kernel of the operator Op$_t^a$ is  
\[ k_t^a(x, y) = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} a(tx + (1-t)y, \xi) d\xi = \frac{1}{(2\pi \hbar)^n} b(tx + (1-t)y, \frac{y-x}{\hbar}), \]
where $b$ is the “partial Fourier transform” of $a$ given by  
\[ b(x, z) = \mathcal{F}_{\xi \rightarrow z}[a(x, \xi)] = \int_{\mathbb{R}^n} e^{-i\xi \cdot z} a(x, \xi) d\xi. \]
Since $a$ is a Schwartz function, $b$ is also a Schwartz function (Check this!). Thus  
\[ \int_{\mathbb{R}^n} |k_t^a(x, y)| dx = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} |b(tx + (1-t)y, \frac{y-x}{\hbar})| dx \]
\[ = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |b(y - t\hbar z, z)| dz \]
\[ \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle z \rangle^{-n-1} dz \cdot \sup_{y, z \in \mathbb{R}^n} |\langle z \rangle^{n+1} b(y, z)| \]
\[ \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle z \rangle^{-n-1} dz \cdot \sup_y \sup_{|\alpha| \leq n+1} \|z^\alpha [\mathcal{F}_{\xi \rightarrow z}[a]](y, z)\|_{L^1(\mathbb{R}_\xi^n)} \]
\[ \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle z \rangle^{-n-1} dz \cdot \sup_y \sup_{|\alpha| \leq n+1} \|\mathcal{F}_{\xi \rightarrow z}(\partial_\xi^\alpha a)(y, z)\|_{L^1(\mathbb{R}_\xi^n)} \]
\[ \leq C_1 := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle z \rangle^{-n-1} dz \cdot \sup_y \sup_{|\alpha| \leq n+1} \|\partial_\xi^\alpha a(y, \xi)\|_{L^1(\mathbb{R}_\xi^n)} \]
and similarly  
\[ \int_{\mathbb{R}^n} |k_t^a(x, y)| dy \leq C_2 := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle z \rangle^{-n-1} dz \cdot \sup_x \sup_{|\alpha| \leq n+1} \|\partial_\xi^\alpha a(x, \xi)\|_{L^1(\mathbb{R}_\xi^n)}. \]
Now the conclusion follows from Schur’s test. \[ \square \]
Remark. One can also prove the $L^2$-boundedness of $\text{Op}_h^t(a)$ directly as follows: We start with Weyl’s decomposition (c.f. the formula (11) in Lecture 9, page 7)

$$(\text{Op}_h^t(a))(x) = \frac{1}{(2\pi \hbar)^{2n}} \int_{\mathbb{R}^{2n}} \left[ \text{Op}_h^t(e^{\frac{i}{\hbar} (y \cdot x + \eta \cdot \xi)}) \right] [(\mathcal{F}_h(s,\xi) \rightarrow (y,\eta))a](y,\eta)dyd\eta.$$ 

Since the operator $\text{Op}_h^t(e^{\frac{i}{\hbar} (y \cdot x + \eta \cdot \xi)})$ is unitary on $L^2(\mathbb{R}^n)$, the triangle inequality implies

$$\|\text{Op}_h^t(a)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \frac{1}{(2\pi \hbar)^{2n}} \int_{\mathbb{R}^{2n}} \|[(\mathcal{F}_h(s,\xi) \rightarrow (y,\eta))a](y,\eta)\|dyd\eta = \frac{1}{(2\pi \hbar)^{2n}} \|\mathcal{F}(s,\xi)\rightarrow (y,\eta)\|_{L^1}.$$ 

It remains to estimate $\|\mathcal{F}(s,\xi)\rightarrow (y,\eta)\|_{L^1}$. Note that here we are using the usual Fourier transform, not the semiclassical one. So our estimate is uniform w.r.t. $\hbar$.

We state and prove a general result:

**Lemma 1.3.** There exists a constant $C = C_n$ such that for any $a \in \mathcal{S}(\mathbb{R}^n)$,

$$\|\mathcal{F}a\|_{L^1} \leq C_n \sup_{\|\alpha\| \leq n+1} \|\partial^\alpha a\|_{L^1}.$$ 

**Proof.** We have

$$\|\mathcal{F}a\|_{L^1} = \int_{\mathbb{R}^n} |\mathcal{F}a(\xi)| d\xi \leq \int_{\mathbb{R}^n} |\langle \xi \rangle^{-n-1} \cdot \langle \xi \rangle^{n+1} \mathcal{F}a(\xi)|_{L^\infty} d\xi \leq C_n \sup_{\|\alpha\| \leq n+1} \|\xi^\alpha \mathcal{F}a\|_{L^\infty} \leq C_n \sup_{\|\alpha\| \leq n+1} \|\partial^\alpha a\|_{L^1}.$$ 

As a consequence, we get:

**Proposition 1.4.** For any $a \in \mathcal{S}(\mathbb{R}^{2n})$, and any $t \in [0,1]$,

$$(2) \quad \|\text{Op}_h^t(a)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C_n \sup_{\|\alpha\| \leq 2n+1} \|\partial^\alpha_{x,\xi} a\|_{L^1}.$$ 

2. $L^2$ BOUNDEDNESS FOR MORE GENERAL SYMBOLS

¶ Boundedness of symbols v.s. $L^2$-boundedness of operators.

We would like to extend the $L^2$-boundedness result we proved above to semi-classical pseudo-differential operators with symbols in more general classes. This is not always possible. For example,

- Neither the operator
  $$Q_j = "\text{multiplication by } x_j"$$
nor the operator

\[ P_j = \frac{\hbar}{i} \frac{\partial}{\partial x_j} \]

is bounded on \( L^2(\mathbb{R}^n) \) (check this statement by providing counterexamples!). This is reasonable since neither the function \( x_j \) (the classical counterpart of \( Q_j \)) nor the function \( \xi_j \) (the classical counterpart of \( P_j \)) are bounded functions.

- On the other hand, if \( a(x) \) is a bounded continuous function, namely \(|a(x)| \leq C\) for all \( x \in \mathbb{R}^n \), then obviously the operator

\[ \text{Op}_\hbar(a) = M_{a(x)} = \text{“multiplication by } a(x)\text{”} \]

is a bounded operator on \( L^2 \) since

\[
\|M_{a(x)}u\|_{L^2} = \left[ \int |a(x)u(x)|^2 dx \right]^{-1/2} \leq C\|u\|_{L^2}.
\]

- Similarly if \( a(\xi) \) is a bounded function, i.e. \(|a(\xi)| \leq C\), then

\[ \text{Op}_\hbar'(a) = F^{-1}_\hbar \circ M_{a(\xi)} \circ F_\hbar \]

is a bounded operator on \( L^2 \), since the Plancherel's Theorem (c.f. Lecture 4, Prop. 1.7) implies

\[
\|\text{Op}_\hbar'(a)u\|_{L^2} = \frac{1}{(2\pi \hbar)^{n/2}} \|M_{a(\xi)}F_\hbar u\|_{L^2} \leq \frac{C}{(2\pi \hbar)^{n/2}} \|F_\hbar u\|_{L^2} = C\|u\|_{L^2}.
\]

\( \Box \) Calderon-Vaillancourt Theorem: Idea of proof.

So one may guess that for any bounded symbol \( a(x, \xi) \), the operator \( \text{Op}_\hbar'(a) \) is a bounded operator on \( L^2 \). Unfortunately this is not quite true in general. (I need an example here.) However, we will prove that if \( a \in S(1) \), namely if we assume the symbol \( a \) itself together with all its derivatives are bounded, then \( \text{Op}_\hbar'(a) \) is bounded on \( L^2(\mathbb{R}^n) \):

**Theorem 2.1** (Calderon-Vaillancourt). If \( a \in S(1) \), then the operator

\[ \text{Op}_\hbar'(a) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \]

is a bounded linear operator on \( L^2(\mathbb{R}^n) \) with

\[
\|\text{Op}_\hbar'(a)\|_{L^2(\mathbb{R}^n)} \leq C \sum_{|\alpha| \leq M_n} h^{|\alpha|/2} \sup_{\mathbb{R}^n} |\partial^\alpha a| \tag{3}
\]

for some universal constant \( M \).\(^1\)

In the rest of this lecture, we prove Calderon-Vaillancourt’s theorem. The idea is the following:

\(^1\)We can take \( Mn \) to be \( 10n + 6 \).
We first decompose $a$ into countably many compactly-supported symbols $a = \sum_m a_m$. This can be done by choosing any partition of unity $1 = \sum_m \chi_m$ such that each $\chi_m$ is compactly supported, and letting $a_m = \chi_m a$. Then formally we have

$$\text{Op}^\dagger_\hbar(a) = \sum_m \text{Op}^\dagger_\hbar(a_m),$$

and by compactness of $\text{supp}(a_m)$, each $\text{Op}^\dagger_\hbar(a_m)$ is $L^2$-bounded.

In general, if $A = \sum A_m$, to conclude the boundedness of $A$ from the boundedness of $A_m$'s,

- a necessary condition we need is that

  the bound of $A_m$'s is uniform for all $m$.

In view of (2), this “uniformly boundedness” can be fulfilled if we choose our partition of unity in a “uniform” way.

- The “uniformly boundedness” of components is still not enough, since there may be “interactions” between different $A_m$’s. Of course the best dream is that if we can choose these $A_m$’s so that there are “no interaction”, or in other words, if these $A_m$’s are “orthogonal” to each other, (namely if the decomposition $A = \sum A_m$ is an “orthogonal decomposition” $A = \bigoplus_m A_m$), then the “uniformly boundedness” of $A_m$’s does imply the boundedness of $A$. But that’s only a dream: we can’t choose our decomposition $\sum_m \text{Op}^\dagger_\hbar(a_m)$ to be orthogonal.

- However, the dream shed a light on the correct direction! The “orthogonality” or the “no-interaction condition” implies that $A_m \circ A_{m'} = 0$ for $m' \neq m$. This is too strong, since it is enough to assume “almost-orthogonality”, or in other words, it is enough if we have a nicely-controlled “interaction”.

- Back to our decomposition. Although we can’t make $\text{Op}^\dagger_\hbar(a_m) \circ \text{Op}^\dagger_\hbar(a_{m'}) = 0$ for all $m' \neq m$, we can make $\text{Op}^\dagger_\hbar(a_m) \circ \text{Op}^\dagger_\hbar(a_{m'}) = O(\hbar^\infty)$ for most $m' \neq m$. In fact, according to Corollary 1.4 in Lecture 9, we have

$$\text{Op}^\dagger_\hbar(a_m) \circ \text{Op}^\dagger_\hbar(a_{m'}) = O(\hbar^\infty)$$

if $\text{supp}(a_m) \cap \text{supp}(a_{m'}) = \emptyset$. In other words, we do have “almost-orthogonality” of $\text{Op}^\dagger_\hbar(a_m)$, if we start with “almost-disjoint” symbols $a_m$’s.[In other words, “almost-orthogonality” is the quantum analogue of the “almost disjointness” of functions].

In summary, here is how we prove the theorem:

A. We first carefully choose a partition of unity $1 = \sum_m \chi_m$, in a uniform way, so that the resulting decomposition $a = \sum (\chi_m a)$ decompose $a$ into a summation of almost disjoint compactly-supported symbols $a_m = \chi_m a$.

B. We then control the interaction between different $\text{Op}^\dagger_\hbar(a_m)$’s.
Finally we add the “almost-orthogonal” $\text{Op}_1^\varepsilon(a_m)$s to get a bounded operator. Technically, this is done by applying the Cotlar-Stein lemma below (which tells us how to “add a sequence of almost-orthogonal bounded operators”):

**Lemma 2.2** (Cotlar-Stein Lemma). Let $H_1, H_2$ be Hilbert spaces. For $j \in \mathbb{N}$ let $A_j \in \mathcal{L}(H_1, H_2)$ be bounded linear operators satisfying

$$\sup_j \sum_{k=1}^{\infty} \|A_j^* A_k\|^{1/2} < C \quad \text{and} \quad \sup_j \sum_{k=1}^{\infty} \|A_j A_k^*\|^{1/2} \leq C.$$ 

The the series $\sum_{j=1}^{\infty} A_j$ converges in the strong operator topology\(^2\) to $A \in \mathcal{L}(H_1, H_2)$ with $\|A\| \leq C$.

A proof will be given in next lecture.

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**Decomposition of symbol.**

As we just explained, we first decompose $a$ into a family of “almost disjoint” compactly supported symbols which are “uniformly controllable”. For this purpose, we will use integer points $\alpha \in \mathbb{R}^d$ as our labels (which are evenly distributed in $\mathbb{R}^d$), and we prove the following “periodic partition of unity”:

**Lemma 2.3.** There exists $\chi_0 \in C_0^\infty(\mathbb{R}^d)$ so that $0 \leq \chi_0 \leq 1$ on $\mathbb{R}^d$, $\text{supp}(\chi_0) \subset B(0, \sqrt{d})$\(^3\) and

$$\sum_{\alpha \in \mathbb{Z}^d} \chi_\alpha = 1 \quad \text{on} \quad \mathbb{R}^d,$$

where for any $\alpha \in \mathbb{Z}^d$, $\chi_\alpha(z) := \chi_0(z - \alpha)$.

**Proof.** First choose $\varphi \in C_0^\infty(\mathbb{R}^d)$ so that $\varphi \geq 0$ on $\mathbb{R}^d$, $\varphi \equiv 1$ on $B(0, \sqrt{d}/2)$ and $\varphi \equiv 0$ on $\mathbb{R}^d \setminus B(0, \sqrt{d})$. Let

$$\psi(z) = \sum_{\alpha \in \mathbb{Z}^d} \varphi(z - \alpha).$$

Then for $z$ in any bounded set, the sum above is a finite sum. Thus $\psi$ is well-defined and is a smooth function. Moreover, by construction,

- for each $z$ one has $\psi(z) \geq 1$ (since for each $z$ one can always find an $\alpha \in \mathbb{Z}^d$ so that $|z_i - \alpha_i| \leq \frac{1}{2}$ for all $i$, i.e. $z - \alpha \in B(0, \sqrt{d}/2)$.)
- for any $\alpha \in \mathbb{Z}^d$, $\psi(z + \alpha) = \psi(z)$.

\(^2\)Recall: the operator strong topology on $\mathcal{L}(H_1, H_2)$ is the topology so that

$$A_j \to A \iff A_j(x) \to A(x) \quad \text{for all} \quad x \in H_1.$$ 

Note that in Cotlar-Stein Lemma, the sum $\sum A_j$ does not converge in operator norm topology.

\(^3\)In Zworski, $\chi_0$ is taken to be supported in $B(0, 2)$, which can’t be true since after translation, these balls can’t cover $\mathbb{R}^{2n}$ for $n > 4$: you need a larger radius to cover points like $(1/2, \cdots, 1/2)$. 

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LECTURE 11: $L^2$-THEORY OF SEMICLASSICAL PSDOS: BOUNDEDNESS

It is easy to see that the function
$$\chi_0(z) = \varphi(z)/\psi(z)$$
is what we want. \hfill □

As a consequence, if we fix such a function $\chi = \chi_0 \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and for any $a = a(x, \xi) \in S(1)$ if we denote
$$a_{\alpha}(x, \xi) = \chi_{\alpha}(x, \xi)a(x, \xi),$$
then we get
$$a(x, \xi) = \sum_{\alpha \in \mathbb{Z}^{2n}} a_{\alpha}(x, \xi).$$
Moreover, these $a_{\alpha}(x, \xi)$ form a countable family of “almost disjoint” compactly-supported symbols such that for any $\beta \in \mathbb{N}^{2n}$, there exists $C_\beta$ (which comes from the bounds of finitely many derivatives of $a \in S(1)$ together with the bounds of finitely many derivatives of $\chi_0$) such that
$$|\partial^\beta a_{\alpha}| \leq C_\beta, \quad \forall \alpha \in \mathbb{Z}^{2n}.$$
As a consequence, there exists $C$ (which depends on $a$) such that for all $\alpha \in \mathbb{Z}^{2n}$,
$$\|\text{Op}_h^t(a_{\alpha})\|_{L(\mathbb{R}^n)} \leq C.$$

“Almost orthogonality”.

In view of the Cotlar-Stein lemma, we need to control the operator norm of $\text{Op}_h^t(a_{\alpha})^* \circ \text{Op}_h^t(a_{\beta})$ and $\text{Op}_h^t(a_{\alpha}) \circ \text{Op}_h^t(a_{\beta})^*$. For simplicity we only consider the case of $t = 1/2$, namely the case of Weyl quantization. The general case can be argued either by a similar proof, or by using the change of quantization formula.

For any $a = a(x, \xi) \in S(1)$, we denote $a_{\alpha} = \chi_{\alpha}a$ as above, and let
$$b_{\alpha\beta} = a_{\alpha} \star a_{\beta},$$
where $\star$ is the Moyal star product. The crucial estimate we need is

Assume $h = 1$.

Lemma 2.4. Suppose $a \in S(1)$. Then for each $N$ and each multi-index $\gamma$, there is a constant
$$C = C(\gamma, N, n) \sum_{|\alpha| \leq 2N+4n+1+|\gamma|} \sup_{\mathbb{R}^n} |\partial^\alpha a|,$$
such that for all $z = (x, \xi) \in \mathbb{R}^{2n}$,
$$|\partial^\gamma b_{\alpha\beta}(z)| \leq C(\alpha - \beta)^{-N} z^{\frac{\alpha+\beta}{2}} - N.$$ (5)

We will not prove this theorem now. Instead, we will prove a stronger version next time: instead of assume $a \in S(1)$, we will only assume $a \in S(m)$ (but the upper bound will also be $m$-dependent).
As a consequence, we get from (2) the following control of \( \| (\hat{b}_{\alpha \beta}^W)_{h=1} \|_{L(\mathbb{L}^2(\mathbb{R}^n))} \) (in which we take \( N = 2n + 1 \) so that \( \langle z - \alpha + \beta \rangle^N \) is in \( L^1 \), and use all \( \gamma \) with \( |\gamma| \leq 2n + 1 \) so that we can apply (2)):

**Corollary 2.5.** For any \( N \) there is a constant 
\[
C = C(n) \sum_{|\alpha| \leq 8n+4} \sup_{\mathbb{R}^n} |\partial^\alpha a|
\]
so that
\[
(6) \quad \| (\hat{b}_{\alpha \beta}^W)_{h=1} \|_{L(\mathbb{L}^2(\mathbb{R}^n))} \leq C \langle \alpha - \beta \rangle^{-2n-1}. 
\]

\[ \blacksquare \text{ Proof of Calderon-Vaiilancourt Theorem.} \]

Finally we finish the proof of Calderon-Vaiilancourt Theorem.

**Proof.**

[Step 1] We first prove a special case: the bound for Weyl quantization with \( h = 1 \):

\[
(7) \quad \| (\hat{a}^W)_{h=1} \|_{L(\mathbb{L}^2(\mathbb{R}^n))} \leq C \sum_{|\alpha| \leq 8n+4} \sup_{\mathbb{R}^n} |\partial^\alpha a|
\]

Set \( A_\alpha = (\hat{a}^W)_{h=1} \), then \( (\hat{b}^W_{\alpha \beta})_{h=1} = A_\alpha^* A_\beta \). By the previous corollary,
\[
\| \hat{b}^W_{\alpha \beta} \|_{L(\mathbb{L}^2)} \leq C \langle \alpha - \beta \rangle^{-2n-1}. 
\]

It follows
\[
\sup_{\alpha} \sum_{\beta} \| A_\alpha A_\beta^* \|^{1/2} \leq C \sum_{\beta} \langle \alpha - \beta \rangle^{-(2n+1)/2} \leq C. 
\]

By the same way one has
\[
\sup_{\alpha} \sum_{\beta} \| A_\alpha^* A_\beta \|^{1/2} \leq C. 
\]

Since \( (\hat{a}^W)_{h=1} = \sum_\alpha A_\alpha \), the conclusion follows from the Cotlar-Stein lemma.

[Step 2] We then prove the theorem for Weyl quantization with general \( h \), i.e.
\[
\| \hat{a}^W \|_{L(\mathbb{L}^2(\mathbb{R}^n))} \leq C \sum_{|\alpha| \leq 8n+4} h^{\frac{|\alpha|}{2}} \sup_{\mathbb{R}^n} |\partial^\alpha a|
\]

This can be done by a re-scaling technique. First we notice that (7) is uniform for all \( h \). In particular, it holds for \( h = 1 \).

Next we introduce the following re-scaling:
\[
\tilde{x} = h^{-1/2} x, \quad \tilde{y} = h^{-1/2} y, \quad \tilde{\xi} = h^{-1/2} \xi
\]
and define
\[
\tilde{u}(\tilde{x}) := u(x) = u(h^{1/2} \tilde{x}), \quad \tilde{\alpha}(\tilde{x}, \tilde{\xi}) := a(x, \xi) = a(h^{1/2} \tilde{x}, h^{1/2} \tilde{\xi}).
\]
One can check: 
\[
\tilde{a}^W u = (\tilde{a})_{h=1} u.
\]
Since the change of variable will create an \( h^{-n/4} \)-factor for \( L^2 \)-norms, namely
\[
\| \tilde{a} \|_{L^2} = h^{-n/4} \| u \|_{L^2}, \quad \text{and} \quad \| \tilde{a}^W u \|_{L^2} = h^{-n/4} \| (\tilde{a})_{h=1} u \|_{L^2}
\]
the conclusion follows from step 1 and
\[
\sup_{\mathbb{R}^n} |\partial_{x,\xi} \tilde{a}| = h^{\lfloor \alpha \rfloor/2} \sup_{\mathbb{R}^n} |\partial^\alpha a|.
\]

[Step 3] Finally we prove the theorem for any \( t \)-quantization. We apply the change of quantization formula (Theorem 3.1 in Lecture 9). Namely, if we set \( b(x, \xi) = e^{i(t-s)hD_x-D_\xi} a(x, \xi) \), then we have \( \text{Op}_h^t(a) = \tilde{b}^W \). It follows
\[
\| \text{Op}_h^t(a) \|_{L^2(\mathbb{R}^n)} = \| \tilde{b}^W \|_{L^2(\mathbb{R}^n)} \leq C \sum_{|\alpha| \leq 8n+4} h^{\lfloor \alpha \rfloor/2} \sup_{\mathbb{R}^{2n}} |\partial^\alpha b|.
\]
We notice that by applying Proposition 1.1 in Lecture 7 to the matrix \( Q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \), we will get
\[
b(x, \xi) = e^{i(t-s)hD_x-D_\xi} a(x, \xi) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\frac{\pi}{2}(t-s) y \cdot \eta} a(x+y, \xi+\eta) dyd\eta.
\]
By inserting \( n+1 \) times before the term \( e^{-i\frac{\pi}{2}(t-s) y \cdot \eta} \) the operator
\[
L = 1 + \sum_j \frac{(D_{\eta_j})^2}{(\frac{1}{2}-t)^2} + \sum_j \frac{(D_{y_j})^2}{(\frac{1}{2}-t)^2}
\]
which satisfies
\[
\frac{1}{(y, \eta)^2} \frac{L}{(t-s)} L(e^{-i\frac{\pi}{2}(t-s) y \cdot \eta}) = e^{-i\frac{\pi}{2}(t-s) y \cdot \eta},
\]
we will get
\[
|\partial^\gamma b| \leq C \sup_{|\rho| \leq |\gamma|+2n+2} |\partial^\rho a|
\]
and thus we conclude
\[
\| \text{Op}_h^t(a) \|_{L^2(\mathbb{R}^n)} \leq C \sum_{|\alpha| \leq 10n+6} h^{\lfloor \alpha \rfloor/2} \sup_{\mathbb{R}^{2n}} |\partial^\alpha a|.
\]

Remark. More generally if \( a \in S_\delta(1) \) for some \( 0 \leq \delta < \frac{1}{2} \), one has
\[
\| \text{Op}_h^t(a) \|_{L^2(\mathbb{R}^n)} \leq C \sum_{|\alpha| \leq M_n} h^{\lfloor \alpha \rfloor/2} \sup_{\mathbb{R}^n} |\partial^\alpha a|.
\]
In particular, for such \( a \) one has
\[
\| \text{Op}_h^t(a) \|_{L^2(\mathbb{R}^n)} \leq C \sup_{\mathbb{R}^{2n}} |a(x, \xi)| + O(h^{1/2-\delta}).
\]