LECTURE 11: L^2-THEORY OF SEMICLASSICAL PsDOs: BOUNDEDNESS

In the previous several lectures, we have studied the definition and basic properties of semiclassical pseudodifferential operators, but mainly as an operator acting on $\mathscr{S}(\mathbb{R}^n)$. However, as we have seen, in quantum part (=the spectral part) of the story the natural space should be a Hilbert space: $\mathscr{S}(\mathbb{R}^n)$ is not. In the next several lectures we shall study properties of semiclassical pseudodifferential operators as linear operators acting on $L^2(\mathbb{R}^n)$, or in cases we need more regularity, acting on the Sobolev spaces $H^s(\mathbb{R}^n)$.

1. L^2-boundedness of $\mathrm{Op}_\hbar^t(a)$ for Schwartz symbols

Suppose $a = a(x, \xi) \in \mathscr{S}(\mathbb{R}^{2n})$ is a Schwartz function. Then as we have seen, the operator $\widehat{a}^W$, or more generally, the operator $\mathrm{Op}_\hbar^t(a)$ for any $t \in [0, 1]$, maps $\mathscr{S}'(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$. In particular, these operators are linear maps from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. It turns out that for a Schwartz symbol, the operator $\mathrm{Op}_\hbar^t(a)$ is always a bounded linear operator (and as we will prove next time, is a compact operator) on $L^2(\mathbb{R}^n)$. In what follows we will provide two different proofs of this fact.

¶ Schur's test.

To prove the L^2-boundedness of linear operators like $\mathrm{Op}_\hbar^t(a)$ which are defined by Schwartz kernels, a very useful criterion is the following Schur's test:

Lemma 1.1 (Schur's Test)**.** *Let $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a continuous function satisfying*

$$C_1 = \sup_x \int_{\mathbb{R}^n} |K(x, y)|dy < +\infty \qquad \text{and} \qquad C_2 = \sup_y \int_{\mathbb{R}^n} |K(x, y)|dx < +\infty,$$

and let A be the linear operator with Schwartz kernel K:

$$Au(x) = \int_{\mathbb{R}^n} K(x, y)u(y)dy.$$

Then A is a bounded linear operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with

$$\|A\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le (C_1C_2)^{\frac{1}{2}} \tag{1}$$

Proof. For any $u \in L^2(\mathbb{R}^n)$ the Cauchy-Schwartz inequality gives

$$|Au(x)|^2 \le \int_{\mathbb{R}^n} |K(x, y)|dy \cdot \int_{\mathbb{R}^n} |K(x, y)||u(y)|^2dy \le C_1 \int_{\mathbb{R}^n} |K(x, y)||u(y)|^2dy.$$

Integrating with respect to x, we get

$$\|Au\|_{L^2}^2 \le \int_{\mathbb{R}^n} \left(C_1 \cdot \int_{\mathbb{R}^n} |K(x,y)||u(y)|^2 dy \right) dx \le C_1 C_2 \|u\|_{L^2}^2.$$

□

¶ L^2-boundedness for PsDOs with Schwartz symbols.

As an immediate consequence,

Theorem 1.2. *If $a = a(x,\xi)$ is a Schwartz function, then for any $t \in [0,1]$,*

$$\mathrm{Op}_\hbar^t(a) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is a bounded linear operator with

$$\|\mathrm{Op}_\hbar^t(a)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le \sup_x \sup_{|\alpha| \le n+1} \|(\partial_\xi^\alpha a)(x,\xi)\|_{L^1(\mathbb{R}^n_\xi)}.$$

Proof. Recall that the Schwartz kernel of the operator $\mathrm{Op}_\hbar^t(a)$ is

$$k_t^a(x,y) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi} a(tx+(1-t)y,\xi)d\xi = \frac{1}{(2\pi\hbar)^n} b(tx+(1-t)y, \frac{y-x}{\hbar}),$$

where b is the "partial Fourier transform" of a given by

$$b(x,z) = \mathcal{F}_{\xi\to z}[a(x,\xi)] = \int_{\mathbb{R}^n} e^{-i\xi\cdot z} a(x,\xi)d\xi.$$

Since a is a Schwartz function, b is also a Schwartz function (Check this!). Thus

$$\begin{aligned}
\int_{\mathbb{R}^n} |k_t^a(x,y)|dx &= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} |b(tx+(1-t)y, \frac{y-x}{\hbar})|dx \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |b(y-t\hbar z, z)|dz \\
&\le \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle z\rangle^{-n-1} dz \cdot \sup_{y,z\in\mathbb{R}^n} |\langle z\rangle^{n+1} b(y,z)| \\
&\le \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle z\rangle^{-n-1} dz \cdot \sup_y \sup_{|\alpha|\le n+1} \|z^\alpha [\mathcal{F}_{\xi\to z} a](y,z)\|_{L^\infty(\mathbb{R}^n_z)} \\
&\le \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle z\rangle^{-n-1} dz \cdot \sup_y \sup_{|\alpha|\le n+1} \|[\mathcal{F}_{\xi\to z}(\partial_\xi^\alpha a)](y,z)\|_{L^\infty(\mathbb{R}^n_z)} \\
&\le C_1 := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle z\rangle^{-n-1} dz \cdot \sup_y \sup_{|\alpha|\le n+1} \|(\partial_\xi^\alpha a)(y,\xi)\|_{L^1(\mathbb{R}^n_\xi)}
\end{aligned}$$

and similarly

$$\int_{\mathbb{R}^n} |k_t^a(x,y)|dy \le C_2 := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle z\rangle^{-n-1} dz \cdot \sup_x \sup_{|\alpha|\le n+1} \|(\partial_\xi^\alpha a)(x,\xi)\|_{L^1(\mathbb{R}^n_\xi)}.$$

Now the conclusion follows from Schur's test. □

Remark. One can also prove the L^2-boundedness of $\mathrm{Op}_\hbar^t(a)$ directly as follows: We start with Weyl's decomposition (c.f. the formula (11) in Lecture 9, page 7)

$$(\mathrm{Op}_\hbar^t(a))(x) = \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} \left[\mathrm{Op}_\hbar^t(e^{\frac{i}{\hbar}(y\cdot x+\eta\cdot\xi)})\right] [(\mathcal{F}_\hbar)_{(s,\xi)\to(y,\eta)} a](y,\eta) dy d\eta.$$

Since the operator $\mathrm{Op}_\hbar^t(e^{\frac{i}{\hbar}(y\cdot x+\eta\cdot\xi)}$ is unitary on $L^2(\mathbb{R}^n)$, the triangle inequality implies

$$\|\mathrm{Op}_\hbar^t(a)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} |[(\mathcal{F}_\hbar)_{(s,\xi)\to(y,\eta)} a](y,\eta)| dy d\eta = \frac{1}{(2\pi)^{2n}} \|\mathcal{F}_{(s,\xi)\to(y,\eta)} a\|_{L^1}.$$

It remains to estimate $\|\mathcal{F}_{(s,\xi)\to(y,\eta)} a\|_{L^1}$. Note that here we are using the usual Fourier transform, not the semiclassical one. So our estimate is uniform w.r.t. $\hbar$. We state and prove a general result:

The L^1-norm of the Fourier transform of a Schwartz function can be controlled by the L^1-norm of its partial derivatives:

Lemma 1.3. *There exists a constant $C = C_n$ such that for any $a \in \mathscr{S}(\mathbb{R}^n)$,*

$$\|\mathcal{F}a\|_{L^1} \le C_n \sup_{\|\alpha\|\le n+1} \|\partial^\alpha a\|_{L^1}.$$

Proof. We have

$$\begin{aligned}\|\mathcal{F}a\|_{L^1} = \int_{\mathbb{R}^n} |\mathcal{F}a(\xi)|\, d\xi &\le \int_{\mathbb{R}^n} \langle\xi\rangle^{-n-1} d\xi \cdot \|\langle\xi\rangle^{n+1}\mathcal{F}a(\xi)\|_{L^\infty} \\ &\le C_n \sup_{|\alpha|\le n+1} \|\xi^\alpha \mathcal{F}a\|_{L^\infty} \le C_n \sup_{\|\alpha\|\le n+1} \|\partial^\alpha a\|_{L^1}.\end{aligned}$$

□

As a consequence, we get:

Proposition 1.4. *For any $a \in \mathscr{S}(\mathbb{R}^{2n})$, and any $t \in [0,1]$,*

$$\|\mathrm{Op}_\hbar^t(a)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C_n \sup_{\|\alpha\|\le 2n+1} \|\partial_{x,\xi}^\alpha a\|_{L^1}. \tag{2}$$

2. L^2 boundedness for more general symbols

¶ Boundedness of symbols v.s. L^2-boundedness of operators.

We would like to extend the L^2-boundedness result we proved above to semiclassical pseudo-differential operators with symbols in more general classes. This is not always possible. For example,

- Neither the operator

$$Q_j = \text{“multiplication by } x_j\text{”}$$

nor the operator

$$P_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j}$$

is bounded on $L^2(\mathbb{R}^n)$ (check this statement by providing counterexamples!). This is reasonable since neither the function x_j (the classical counterpart of Q_j) nor the function ξ_j (the classical counterpart of P_j) are bounded functions.

- On the other hand, if $a(x)$ is a bounded continuous function, namely $|a(x)| \le C$ for all $x \in \mathbb{R}^n$, then obviously the operator

$$\mathrm{Op}_\hbar^t(a) = M_{a(x)} = \text{“multiplication by } a(x)\text{”}$$

is a bounded operator on L^2 since

$$\|M_{a(x)}u\|_{L^2} = \left[\int |a(x)u(x)|^2 dx\right]^{-1/2} \le C\|u\|_{L^2}.$$

- Similarly if $a(\xi)$ is a bounded function, i.e. $|a(\xi)| \le C$, then

$$\mathrm{Op}_\hbar^t(a) = \mathcal{F}_\hbar^{-1} \circ M_{a(\xi)} \circ \mathcal{F}_\hbar$$

is a bounded operator on L^2, since the Plancherel's Theorem (c.f. Lecture 4, Prop. 1.7) implies

$$\|\mathrm{Op}_\hbar^t(a)u\|_{L^2} = \frac{1}{(2\pi\hbar)^{n/2}}\|M_{a(\xi)}\mathcal{F}_\hbar u\|_{L^2} \le \frac{C}{(2\pi\hbar)^{n/2}}\|\mathcal{F}_\hbar u\|_{L^2} = C\|u\|_{L^2}.$$

¶ Calderon-Vaillancourt Theorem: Idea of proof.

So one may guess that for any bounded symbol $a(x,\xi)$, the operator $\mathrm{Op}_\hbar^t(a)$ is a bounded operator on L^2. Unfortunately this is not quite true in general. (I need an example here.) However, we will prove that if $a \in S(1)$, namely if we assume the symbol a itself together with all its derivatives are bounded, then $\mathrm{Op}_\hbar^t(a)$ is bounded on $L^2(\mathbb{R}^n)$:

Theorem 2.1 (Calderon-Vaillancourt)**.** *If $a \in S(1)$, then the operator*

$$\mathrm{Op}_\hbar^t(a) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is a bounded linear operator on $L^2(\mathbb{R}^n)$ with

$$\|\mathrm{Op}_\hbar^t(a)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C \sum_{|\alpha|\le Mn} \hbar^{|\alpha|/2} \sup_{\mathbb{R}^{2n}} |\partial^\alpha a| \tag{3}$$

for some universal constant M.[1]

In the rest of this lecture, we prove Calderon-Vaillancourt's theorem. The idea is the following:

[1] We can take Mn to be $10n+6$.

- We first decompose a into countably many compactly-supported symbols $a = \sum_m a_m$. This can be done by choosing any partition of unity $1 = \sum_m \chi_m$ such that each χ_m is compactly supported, and letting $a_m = \chi_m a$. Then formally we have
$$\mathrm{Op}_\hbar^t(a) = \sum_m \mathrm{Op}_\hbar^t(a_m),$$
and by compactness of $\mathrm{supp}(a_m)$, each $\mathrm{Op}_\hbar^t(a_m)$ is L^2-bounded.
- In general, if $A = \sum A_m$, to conclude the boundedness of A from the boundedness of A_m's,
  - a necessary condition we need is that
  $$\text{the bound of } A_m\text{'s is uniform for all } m.$$
  In view of (2), this "uniformly boundedness" can be fulfilled if we choose our partition of unity in a "uniform" way.
  - The "uniformly boundedness" of components is still not enough, since there may be "interactions" between different A_m's. Of course the best dream is that if we can choose these A_m's so that there are "no interaction", or in other words, if these A_m's are "orthogonal" to each other, (namely if the decomposition $A = \sum_m A_m$ is an "orthogonal decomposition" $A = \bigoplus_m A_m$), then the "uniformly boundedness" of A_m's does imply the boundedness of A. But that's only a dream: we can't choose our decomposition $\sum_m \mathrm{Op}_\hbar^t(a_m)$ to be orthogonal.
  - However, the dream shed a light on the correct direction! The "orthogonality" or the "no-interaction condition" implies that $A_m \circ A_{m'} = 0$ for $m' \neq m$. This is too strong, since it is enough to assume "almost-orthogonality", or in other words, it is enough if we have a nicely-controlled "interaction".
- Back to our decomposition. Although we can't make $\mathrm{Op}_\hbar^t(a_m) \circ \mathrm{Op}_\hbar^t(a_{m'}) = 0$ for <u>all</u> $m' \neq m$, we can make $\mathrm{Op}_\hbar^t(a_m) \circ \mathrm{Op}_\hbar^t(a_{m'}) = O(\hbar^\infty)$ for <u>most</u> $m' \neq m$. In fact, according to Corollary 1.4 in Lecture 9, we have
$$\mathrm{Op}_\hbar^t(a_m) \circ \mathrm{Op}_\hbar^t(a_{m'}) = O(\hbar^\infty)$$
if $\mathrm{supp}(a_m) \cap \mathrm{supp}(a_{m'}) = \emptyset$. In other words, we do have "almost-orthogonality" of $\mathrm{Op}_\hbar^t(a_m)$, if we start with "almost-disjoint" symbols a_m's.[In other words, "almost-orthogonality" is the quantum analogue of the "almost disjointness" of functions].

In summary, here is how we prove the theorem:

<u>A</u> We first carefully choose a partition of unity $1 = \sum_m \chi_m$, in a uniform way, so that the resulting decomposition $a = \sum(\chi_m a)$ decompose a into a summation of almost disjoint compactly-supported symbols $a_m = \chi_m a$.

<u>B</u> We then control the interaction between different $\mathrm{Op}_\hbar^t(a_m)$s.

C Finally we add the "almost-orthogonal" $\mathrm{Op}_h^t(a_m)$s to get a bounded operator. Technically, this is done by applying the Cotlar-Stein lemma below (which tells us how to "add a sequence of almost-orthogonal bounded operators"):

Lemma 2.2 (Cotlar-Stein Lemma)**.** *Let H_1, H_2 be Hilbert spaces. For $j \in \mathbb{N}$ let $A_j \in \mathcal{L}(H_1, H_2)$ be bounded linear operators satisfying*

$$\sup_j \sum_{k=1}^{\infty} \|A_j^* A_k\|^{1/2} < C \qquad \textit{and} \qquad \sup_j \sum_{k=1}^{\infty} \|A_j A_k^*\|^{1/2} \le C.$$

The the series $\sum_{j=1}^{\infty} A_j$ converges in the strong operator topology[2] *to $A \in \mathcal{L}(H_1, H_2)$ with $\|A\| \le C$.*

A proof will be given in next lecture.

¶ Decomposition of symbol.

As we just explained, we first decompose a into a family of "almost disjoint" compactly supported symbols which are "uniformly controllable". For this purpose, we will use integer points $\alpha \in \mathbb{R}^d$ as our labels (which are evenly distributed in $\mathbb{R}^d$), and we prove the following "periodic partition of unity":

Lemma 2.3. *There exists $\chi_0 \in C_0^\infty(\mathbb{R}^d)$ so that $0 \le \chi_0 \le 1$ on $\mathbb{R}^d$,* $\mathrm{supp}(\chi_0) \subset B(0, \sqrt{d})$ [3] *and*

$$\sum_{\alpha \in \mathbb{Z}^d} \chi_\alpha = 1 \qquad \textit{on} \quad \mathbb{R}^d, \tag{4}$$

where for any $\alpha \in \mathbb{Z}^d$, $\chi_\alpha(z) := \chi_0(z - \alpha)$.

Proof. First choose $\varphi \in C_0^\infty(\mathbb{R}^d)$ so that $\varphi \ge 0$ on $\mathbb{R}^d$, $\varphi \equiv 1$ on $B(0, \sqrt{d}/2)$ and $\varphi \equiv 0$ on $\mathbb{R}^d \setminus B(0, \sqrt{d})$. Let

$$\psi(z) = \sum_{\alpha \in \mathbb{Z}^d} \varphi(z - \alpha).$$

Then for z in any bounded set, the sum above is a finite sum. Thus ψ is well-defined and is a smooth function. Moreover, by construction,

- for each z one has $\psi(z) \ge 1$ (since for each z one can always find an $\alpha \in \mathbb{Z}^d$ so that $|z_i - \alpha_i| \le \frac{1}{2}$ for all i, i.e. $z - \alpha \in \overline{B(0, \sqrt{d}/2)}$.)
- for any $\alpha \in \mathbb{Z}^d$, $\psi(z + \alpha) = \psi(z)$.

[2] Recall: the operator strong topology on $\mathcal{L}(H_1, H_2)$ is the topology so that

$$A_j \to A \Longleftrightarrow A_j(x) \to A(x) \quad \text{for all} \quad x \in H_1.$$

Note that in Cotlar-Stein Lemma, the sum $\sum A_j$ does not converge in operator norm topology.

[3] In Zworski, χ_0 is taken to be supported in $B(0, 2)$, which can't be true since after translation, these balls can't cover $\mathbb{R}^{2n}$ for $n > 4$: you need a larger radius to cover points like $(1/2, \cdots, 1/2)$.

It is easy to see that the function

$$\chi_0(z) = \varphi(z)/\psi(z)$$

is what we want. □

As a consequence, if we fix such a function $\chi = \chi_0 \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and for any $a = a(x,\xi) \in S(1)$ if we denote

$$a_\alpha(x,\xi) = \chi_\alpha(x,\xi)a(x,\xi),$$

then we get

$$a(x,\xi) = \sum_{\alpha\in\mathbb{Z}^{2n}} a_\alpha(x,\xi).$$

Moreover, these $a_\alpha(x,\xi)$ form a countable family of "almost disjoint" compactly-supported symbols such that for any $\beta \in \mathbb{N}^{2n}$, there exists C_β (which comes from the bounds of finitely many derivatives of $a \in S(1)$ together with the bounds of finitely many derivatives of χ_0) such that

$$|\partial^\beta a_\alpha| \le C_\beta, \quad \forall \alpha \in \mathbb{Z}^{2n}.$$

As a consequence, there exists C (which depends on a) such that for all $\alpha \in \mathbb{Z}^{2n}$,

$$\|\mathrm{Op}_\hbar^t(a_\alpha)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C.$$

¶ "Almost orthogonality".

In view of the Cotlar-Stein lemma, we need to control the operator norm of $\mathrm{Op}_\hbar^t(a_\alpha)^* \circ \mathrm{Op}_\hbar^t(a_\beta)$ and $\mathrm{Op}_\hbar^t(a_\alpha) \circ \mathrm{Op}_\hbar^t(a_\beta)^*$. For simplicity we only consider the case of $t = 1/2$, namely the case of Weyl quantization. The general case can be argued either by a similar proof, or by using the change of quantization formula.

For any $a = a(x,\xi) \in S(1)$, we denote $a_\alpha = \chi_\alpha a$ as above, and let

$$b_{\alpha\beta} = \bar{a}_\alpha \star a_\beta,$$

where $\star$ is the Moyal star product. The crucial estimate we need is

Assume $\hbar = 1$.

Lemma 2.4. *Suppose $a \in S(1)$. Then for each N and each multi-index γ, there is a constant*

$$C = C(\gamma, N, n) \sum_{|\alpha|\le 2N+4n+1+|\gamma|} \sup_{\mathbb{R}^n} |\partial^\alpha a|,$$

such that for all $z = (x,\xi) \in \mathbb{R}^{2n}$,

$$|\partial^\gamma b_{\alpha\beta}(z)| \le C\langle \alpha-\beta\rangle^{-N}\langle z - \frac{\alpha+\beta}{2}\rangle^{-N}. \tag{5}$$

We will not prove this theorem now. Instead, we will prove a stronger version next time: instead of assume $a \in S(1)$, we will only assume $a \in S(m)$ (but the upper bound will also be m-dependent).

As a consequence, we get from (2) the following control of $\|(\widehat{b_{\alpha\beta}}^W)_{\hbar=1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$ (in which we take $N=2n+1$ so that $\langle z-\frac{\alpha+\beta}{2}\rangle^N$ is in L^1, and use all γ with $|\gamma|\le 2n+1$ so that we can apply (2)):

Corollary 2.5. *For any N there is a constant*

$$C = C(n)\sum_{|\alpha|\le 8n+4}\sup_{\mathbb{R}^n}|\partial^\alpha a|$$

so that

$$\|(\widehat{b_{\alpha\beta}}^W)_{\hbar=1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\le C\langle\alpha-\beta\rangle^{-2n-1}. \tag{6}$$

¶ Proof of Calderon-Vailancourt Theorem.

Finally we finish the proof of Calderon-Vailancourt Theorem.

Proof.

Step 1 We first prove a special case: the bound for Weyl quantization with $\hbar=1$:

$$\|(\widehat{a}^W)_{\hbar=1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\le C\sum_{|\alpha|\le 8n+4}\sup_{\mathbb{R}^{2n}}|\partial^\alpha a| \tag{7}$$

Set $A_\alpha=(\widehat{a}^W)_{\hbar=1}$, then $(\widehat{b}^W_{\alpha\beta})_{\hbar=1}=A_\alpha^*A_\beta$. By the previous corollary,

$$\|\widehat{b}^W_{\alpha\beta}\|_{\mathcal{L}(L^2)}\le C\langle\alpha-\beta\rangle^{-2n-1}.$$

It follows

$$\sup_\alpha\sum_\beta\|A_\alpha A_\beta^*\|^{1/2}\le C\sum_\beta\langle\alpha-\beta\rangle^{-(2n+1)/2}\le C.$$

By the same way one has

$$\sup_\alpha\sum_\beta\|A_\alpha^*A_\beta\|^{1/2}\le C.$$

Since $(\widehat{a}^W)_{\hbar=1}=\sum_\alpha A_\alpha$, the conclusion follows from the Cotlar-Stein lemma.

Step 2 We then prove the theorem for Weyl quantization with general $\hbar$, i.e.

$$\|\widehat{a}^W\|_{\mathcal{L}(L^2(\mathbb{R}^n))}\le C\sum_{|\alpha|\le 8n+4}\hbar^{|\alpha|/2}\sup_{\mathbb{R}^{2n}}|\partial^\alpha a|$$

This can be done by a re-scaling technique. First we notice that (7) is uniform for all $\hbar$. In particular, it holds for $\hbar=1$.

Next we introduce the following re-scaling:

$$\tilde{x}=\hbar^{-1/2}x,\quad \tilde{y}=\hbar^{-1/2}y,\quad \tilde{\xi}=\hbar^{-1/2}\xi$$

and define

$$\tilde{u}(\tilde{x}):=u(x)=u(\hbar^{1/2}\tilde{x}),\quad \tilde{a}(\tilde{x},\tilde{\xi}):=a(x,\xi)=a(\hbar^{1/2}\tilde{x},\hbar^{1/2}\tilde{\xi}).$$

One can check:

$$\widehat{a}^W u = (\widehat{\tilde{a}}^W)_{\hbar=1}\tilde{u}.$$

Since the change of variable will create an $\hbar^{-n/4}$-factor for L^2-norms, namely

$$\|\tilde{u}\|_{L^2} = \hbar^{-n/4}\|u\|_{L^2}, \quad \text{and} \quad \|\widehat{a}^W u\|_{L^2} = \hbar^{-n/4}\|(\widehat{\tilde{a}}^W)_{\hbar=1}\tilde{u}\|_{L^2}$$

the conclusion follows from step 1 and

$$\sup_{\mathbb{R}^n}|\partial_{\tilde{x},\tilde{\xi}}\tilde{a}| = \hbar^{|\alpha|/2}\sup_{\mathbb{R}^n}|\partial^\alpha a|.$$

Step 3 Finally we prove the theorem for any t-quantization. We apply the change of quantization formula (Theorem 3.1 in Lecture 9). Namely, if we set $b(x,\xi) = e^{i(t-\frac{1}{2})\hbar D_x\cdot D_\xi}a(x,\xi)$, then we have $\mathrm{Op}_\hbar^t(a) = \widehat{b}^W$. It follows

$$\|\mathrm{Op}_\hbar^t(a)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = \|\widehat{b}^W\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C\sum_{|\alpha|\le 8n+4}\hbar^{|\alpha|/2}\sup_{\mathbb{R}^{2n}}|\partial^\alpha b|$$

. We notice that by applying Proposition 1.1 in Lecture 7 to the matrix $Q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, we will get

$$b(x,\xi) = e^{i(t-s)\hbar D_x\cdot D_\xi}a(x,\xi) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}(t-\frac{1}{2})y\cdot\eta}a(x+y,\xi+\eta)dyd\eta.$$

By inserting $n+1$ times before the term $e^{-\frac{i}{\hbar}(t-\frac{1}{2})y\cdot\eta}$ the operator

$$L = 1 + \sum_j \frac{(D_{\eta_j})^2}{(\frac{1}{2}-t)^2} + \sum_j \frac{(D_{y_j})^2}{(\frac{1}{2}-t)^2}$$

which satisfies

$$\frac{1}{\langle y,\eta\rangle^2}L(e^{-\frac{i}{\hbar}(t-\frac{1}{2})y\cdot\eta}) = e^{-\frac{i}{\hbar}(t-\frac{1}{2})y\cdot\eta},$$

we will get

$$|\partial^\gamma b| \le C\sup_{|\rho|\le|\gamma|+2n+2}|\partial^\rho a|$$

and thus we conclude

$$\|\mathrm{Op}_\hbar^t(a)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C\sum_{|\alpha|\le 10n+6}\hbar^{|\alpha|/2}\sup_{\mathbb{R}^{2n}}|\partial^\alpha a|.$$

□

Remark. More generally if $a \in S_\delta(1)$ for some $0 \le \delta < \frac{1}{2}$, one has

$$\|\mathrm{Op}_\hbar^t(a)\|_{\mathcal{L}(L^2)} \le C\sum_{|\alpha|\le Mn}\hbar^{|\alpha|/2}\sup_{\mathbb{R}^n}|\partial^\alpha a|.$$

In particular, for such a one has

$$\|\mathrm{Op}_\hbar^t(a)\|_{\mathcal{L}(L^2)} \le C\sup_{\mathbb{R}^{2n}}|a(x,\xi)| + O(\hbar^{1/2-\delta}).$$