LECTURE 13: L^2-THEORY OF SEMICLASSICAL PsDOs: HILBERT-SCHMIDT AND TRACE CLASS OPERATORS

In today's lecture we start with some abstract properties of Hilbert-Schmidt operators and trace class operators. Then we will investigate the following natural questions: for which class of symbols, the quantized operator $\mathrm{Op}_\hbar^t(a)$ is a Hilbert-Schmidt or trace class operator? How to compute its trace?

1. Hilbert-Schmidt and Trace class operators: Abstract theory

¶ Definitions via singular values.

As we have mentioned last time, we are aiming at developing the spectral theory for semiclassical pseudodifferential operators. For $a \in S(m)$, where m is a decaying symbol, we showed that $\mathrm{Op}_\hbar^t(a)$ is a compact operator acting on $L^2(\mathbb{R}^n)$. As a consequence, either $\mathrm{Op}_\hbar^t(a)$ has only finitely many nonzero eigenvalues, or it has infinitely nonzero eigenvalues which can be arranged to a sequence whose norms converges to 0.

However, in general the eigenvalues of a compact operator A are non-real. A very simple way to get real eigenvalues is to consider the operator A^*A, which is a compact self-adjoint linear operator acting on $L^2(\mathbb{R}^n)$. Thus the eigenvalues [1] of A^*A can be list[2] in decreasing order as

$$s_1^2 \geq s_2^2 \geq s_3^2 \geq \cdots .$$

The numbers s_j (which will always be taken to be the positive one) are called the *singular values* of A.

Definition 1.1. We say a compact operator[3] A is a *Hilbert-Schmidt* operator if

$$\|A\|_{HS} := \left(\sum_j s_j^2\right)^{1/2} < +\infty. \tag{1}$$

The number $\|A\|_{HS}$ is called the *Hilbert-Schmidt norm* (or *Schatten 2-norm*) of A.

Another very important class of operators for us is the trace class operators.

[1]Note that by definition, the eigenvalues of A^*A are nonnegative.

[2]We always repeat an eigenvalue according to its multiplicity.

[3]The "official" definition of Hilbert-Schmidt operator assumes only the boundedness of A. However, it is easy to prove that if A is Hilbert-Schmidt operator, then it can be written as the limit (with respect to the operator norm topology) of a sequence of finite rank operators, and thus must be a compact operator.

Definition 1.2. We say a compact operator A is a *trace class* operator if

$$\|A\|_{tr} := \sum_j s_j < +\infty. \tag{2}$$

The number $\|A\|_{tr}$ is called the *trace norm* (or the *Schatten 1-norm*) of A.

¶ Abstract definitions via Hilbert basis.

In general the singular values of an operator are very hard to compute. Fortunately, we have an alternative characterization of Hilbert-Schmidt norm (and thus Hilbert-Schmidt operators) via Hilbert bases, which is easier to use. Let H be a separable Hilbert space, and $A \in \mathcal{L}(H)$ is a bounded linear operator. If $\{e_i\}, \{f_i\}$ are two orthonormal bases of H, then by the Parseval's identity,

$$\sum_i \|Ae_i\|^2 = \sum_i \sum_j |\langle Ae_i, f_j\rangle|^2 = \sum_{i,j} |\langle A^* f_j, e_i\rangle|^2 = \sum_j \|A^* f_j\|^2.$$

As a consequence, the quantity

$$\sum_j \|Ae_j\|^2 = \sum_j \|A^* f_j\|^2 = \sum_j \|A\tilde{e}_j\|^2$$

is independent of the choice of the orthonormal basis $\{e_i\}$. Moreover, one can prove that if the above quantity is finite, then A must be a compact operator. In particular, if we take e_n to be the orthonormal basis consisting of eigenvectors of A^*A (whose existence is guaranteed by the spectral theory of self-adjoint compact operators), then

$$\sum_j \|Ae_j\|^2 = \sum_j \langle Ae_j, Ae_j\rangle = \sum_j \langle A^*Ae_j, e_j\rangle = \sum_j s_j^2.$$

So we get the following alternative characterization of Hilbert-Schmidt norm:

$$\|A\|_{HS} = \left(\sum_j \|Ae_j\|^2\right)^{1/2}, \tag{3}$$

where $\{e_i\}$ is any basis of H.

Similarly we can define trace class operators by using a basis of H: For $A \in \mathcal{K}(H)$. Then one can define a positive self-adjoint operator $|A| \in \mathcal{L}(H)$ by

$$|A|^2 = A^*A.$$

Applying the previous computation to $|A|$, we see that the quantity

$$\sum_j \langle |A|e_j, e_j\rangle = \sum_j \||A|^{1/2}e_j\|^2$$

is independent of the choice of $\{e_n\}$. If we take $\{e_n\}$ to be the orthonormal basis consisting of eigenvectors of A^*A, then

$$\sum_j \langle |A|e_j, e_j\rangle = \sum_j s_j.$$

In other words, we get the following alternative expression of the trace norm:

$$\|A\|_{tr} = \sum_j \langle |A| e_j, e_j \rangle. \tag{4}$$

¶ Spaces of operators v.s. spaces of sequences.

We will denote the space of Hilbert-Schmidt operators on H by $\mathcal{L}_2(H)$, and denote the space of trace class operators on H by $\mathcal{L}_1(H)$. It turns out that they can be viewed as a "non-commutative analogues/generalizations" of the spaces l^2 and l^1, and have very similar properties. Here is an interesting table comparing spaces of sequences and spaces of operators on a separable Hilbert space H:

	space of sequences	space of operators
subspace	c_{00} (eventually zero sequences)	$\mathcal{F}(H)$ (finite rank operators)
Banach Spaces	l^1 (summable sequences)	$\mathcal{L}_1(H)$ (trace class operators)
Hilbert Spaces	l^2 (square summable sequences)	$\mathcal{L}_2(H)$ (Hilbert-Schmidt operators)
Banach Spaces	c_0 (sequences converge to 0)	$\mathcal{K}(H)$ (compact operators)
Banach Spaces	l^∞ (bounded sequences)	$\mathcal{L}(H)$ (bounded operators)
inclusion	$c_{00} \subset l^1 \subset l^2 \subset c_0 \subset l^\infty$	$\mathcal{F}(H) \subset \mathcal{L}_1(H) \subset \mathcal{L}_2(H) \subset \mathcal{K}(H) \subset \mathcal{L}(H)$
dense	c_{00} is dense in $(l^1, \|\cdot\|_{l^1})$, in $(l^2, \|\cdot\|_{l^2})$ and in $(l^\infty, \|\cdot\|_{l^\infty})$	$\mathcal{F}(H)$ is dense in $(\mathcal{L}_1(H), \|\cdot\|_{tr})$, in $(\mathcal{L}_2(H), \|\cdot\|_{HS})$ and in $(\mathcal{K}(H), \|\cdot\|_{\mathcal{L}})$
norm	$\|\cdot\|_{l^\infty} \leq \|\cdot\|_{l^2} \leq \|\cdot\|_{l^1}$	$\|\cdot\|_{\mathcal{L}} \leq \|\cdot\|_{HS} \leq \|\cdot\|_{tr}$
Duality	$(l^1)^* = l^\infty$	$(\mathcal{L}_1(H))^* = \mathcal{L}(H)$
Duality	$(c_0)^* = l^1$	$(\mathcal{K}(H))^* = \mathcal{L}_1(H)$
ideal	$l^\infty l^1, l^1 l^\infty \subset l^1$	$\mathcal{L}_1(H)\mathcal{L}(H), \mathcal{L}(H)\mathcal{L}_1(H) \subset \mathcal{L}_1(H)$
ideal	$l^\infty l^2, l^2 l^\infty \subset l^2$	$\mathcal{L}_2(H)\mathcal{L}(H), \mathcal{L}(H)\mathcal{L}_2(H) \subset \mathcal{L}_2(H)$
decomposition	$l^1 = l^2 l^2$	$\mathcal{L}_1(H) = \mathcal{L}_2(H)\mathcal{L}_2(H)$
decomposition	$\|(a_n b_n)\|_{l^1} \leq \|(a_n)\|_{l^2} \|(b_n)\|_{l^2}$	$\|AB\|_{tr} \leq \|A\|_{HS} \|B\|_{HS}$

For proofs of these properties, we refer to Reed-Simon, Volume 1, §6.6.

¶ The trace of trace class operators.

For trace class operators, one can define a linear functional called the *trace*. It will play the same role as the trace for matrices that we learned in linear algebra. In fact, one of our ultimate goals is to study the traces of certain semiclassical pseudodifferential operators.

On the space of trace class operators, we can define a *trace functional* via

$$\mathrm{Tr} : \mathcal{L}_1(H) \to \mathbb{C}, \quad A \mapsto \mathrm{Tr}(A) = \sum_j \langle Ae_j, e_j \rangle.$$

Although it is not that obvious, the quantity above *is* independent of the choices of $\{e_j\}$: to see this one start with the "polar decomposition" $A = U|A|$ of A, where U is an isometric when restricted to the closed subspace $\mathrm{Ker}(U)^\perp$ (For the existence of polar decomposition, c.f. Reed-Simon, §6.4). Then

$$\begin{aligned}\sum_j \langle Ae_j, e_j \rangle = \sum_j \langle U|A|e_j, e_j \rangle &= \sum_j \langle |A|^{1/2} e_j, |A|^{1/2} U^* e_j \rangle \\ &= \sum_j \langle |A|^{1/2} e_j, f_j \rangle \overline{\langle |A|^{1/2} U^* e_j, f_j \rangle} \\ &= \sum_j \overline{\langle |A|^{1/2} f_j, e_j \rangle} \langle U|A|^{1/2} f_j, e_j \rangle \\ &= \sum_j \langle U|A|^{1/2} f_j, |A|^{1/2} f_j \rangle\end{aligned}$$

and the conclusion follows. Note that by definition, for any $A \in \mathcal{L}_2(H)$, we have $\|A\|_{HS}^2 = \mathrm{Tr}(A^*A)$.

We list without proof a couple basis properties of the trace functional:

- The trace functional is continuous (with respect to the trace norm):
$$|\mathrm{Tr}(A)| \le \|A\|_{tr}.$$
- By definition, if A is trace class and self-adjoint, then
$$\mathrm{Tr}(A) = \sum_j \lambda_j,$$
where λ_js are eigenvalues of A (counting multiplicity). It turns out that the formula holds for any trace class operator (but the proof is more involved):

  Theorem 1.3 (Lidskii)**.** *Suppose A is of trace class, and*
  $$\mathrm{Spec}(A) = \{\lambda_j\}, \quad |\lambda_1| \ge |\lambda_2| \ge \cdots \to 0,$$
  then $\mathrm{Tr}(A) = \sum_j \lambda_j$.
- For any $A \in \mathcal{L}_1(H)$ and $B \in \mathcal{L}(H)$, we have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

2. Hilbert-Schmidt and trace class integral operators

¶ Hilbert-Schmidt integral operators.

A very important class of Hilbert-Schmidt operators are *Hilbert-Schmidt integral operators*, which are by definition Hilbert-Schmidt operators on L^2 spaces of the form

$$A = A_K : \varphi \mapsto [A_K\varphi](x) = \int_{\mathbb{R}^n} K(x,y)\varphi(y)dy.$$

(Of course in the definition of Hilbert-Schmidt integral operators, one may replace $\mathbb{R}^n$ by any measure space.)

Let $K = K(x,y)$ be a measurable function defined on $\mathbb{R}^n_x \times \mathbb{R}^n_y$. We want to find out conditions so that the integral operator A_K is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$. For this purpose we fix $x \in \mathbb{R}^n_x$ and denote $K_x(y) := K(x,y)$. Then by definition, for any $\varphi \in L^2(\mathbb{R}^n)$,

$$A_K\varphi(x) = \langle K_x, \overline{\varphi}\rangle_{L^2(\mathbb{R}^n_y)}.$$

So, if we want $A_K\varphi(x)$ to be a well-defined measurable function, we need to require $K_x \in L^2(\mathbb{R}^n_y)$. Of course we want more: we want $A_K\varphi \in L^2(\mathbb{R}^n)$. For this purpose we calculate via Cauchy-Schwarz inequality:

$$\begin{aligned}\|A_K\varphi\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |[A_K\varphi](x)|^2 dx &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} K(x,y)u(y)dy \right|^2 dx \\ &\le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |K(x,y)|^2 dy \right) \left(\int_{\mathbb{R}^n} |u(y)|^2 dy \right) dx \\ &= \|K\|^2_{L^2(\mathbb{R}^{2n})} \cdot \|\varphi\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

So the condition we need is $K(x,y) \in L^2(\mathbb{R}^n_x \times \mathbb{R}^n_y)$. Note that if this is satisfied, then by Fubini's theorem, $K_x(y) := K(x,y) \in L^2(\mathbb{R}^n_y)$ for a.e. $x \in \mathbb{R}^n$, so our first requirement is satisfied automatically. In summary, we have shown:

> Fact 1. For A_K to be a linear map from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, we need to require $K = K(x,y) \in L^2(\mathbb{R}^n_x \times \mathbb{R}^n_y)$. Moreover, under this assumption, A_K is in fact a bounded linear map with

(5) $$\|A_K\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le \|K\|_{L^2(\mathbb{R}^{2n})}.$$

To get more information of the operator A_K, we may try to approximate A_K by simpler operators. By definition, to approximate A_K, it is enough to approximate the kernel function K. So we let $\{\varphi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$. Then $\{\varphi_j(x)\overline{\varphi_k(y)}\}_{j,k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and we can decompose

$$K(x,y) = \sum_{j,k} c_{jk}\varphi_j(x)\overline{\varphi_k(y)}.$$

Now the kernel function K is naturally approximated by the "truncated" kernel

$$K_N(x,y) := \sum_{j,k\le N} c_{jk}\varphi_j(x)\overline{\varphi_k(y)}.$$

Since the operator

$$A_{K_N} = \sum_{j,k\le N} c_{jk}\langle\cdot,\varphi_k\rangle\varphi_j$$

is a finite rank operator and by (5),

$$\|A_K - A_{K_N}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le \|K_N - K\|_{L^2(\mathbb{R}^n)} \to 0$$

as $N\to\infty$, we immediately deduce that A_K is a compact operator on $L^2(\mathbb{R}^n)$ as long as the kernel $K = K(x,y) \in L^2(\mathbb{R}^n_x \times \mathbb{R}^n_y)$. But of course this is not our final goal. We want A_K to be Hilbert-Schmidt operator. For this purpose we calculate the Hilbert-Schmidt norm of A_K via the basis $\{\varphi_k\}$:

$$\|A_K\|_{HS}^2 = \sum_l \|A_K\varphi_l\|^2_{L^2(\mathbb{R}^n)} = \sum_{j,k,l} \|c_{jk}\varphi_j(x)\delta_{kl}\|^2_{L^2(\mathbb{R}^n_x)} = \sum_{j,l}|c_{jl}|^2 = \|K\|^2_{L^2(\mathbb{R}^{2n})}.$$

So we conclude:

Fact 2. For any $K = K(x,y) \in L^2(\mathbb{R}^{2n})$, the integral operator A_K is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$ with

$$\|A_K\|_{HS} = \|K\|_{L^2(\mathbb{R}^{2n})}. \tag{6}$$

We can conclude more: Consider the map

$$\mathcal{A} : L^2(\mathbb{R}^{2n}) \to \mathcal{L}^2(L^2(\mathbb{R}^n)), \quad K \mapsto A_K.$$

Then by (6), $\mathcal{A}$ is an isometric embedding. In particular, the image of $\mathcal{A}$ is closed. On the other hand, any finite rank operator on $L^2(\mathbb{R}^n)$ is an integral operator with L^2-kernel. Proof: If A is a finite rank operator, i.e. $\mathrm{Im}(A)$ is finitely dimensional, then after fixing a basis $\psi_1,\cdots,\psi_m$ of $\mathrm{Im}(A)$, we can write A uniquely as

$$A\varphi = c_1(\varphi)\psi_1 + \cdots + c_m(\varphi)\psi_m.$$

The linearity of A implies that each $c_j(\varphi)$ is linear functional on $L^2(\mathbb{R}^n)$. By Riesz representation theorem, there exists φ_j such that $c_j(\varphi) = \langle\varphi,\varphi_j\rangle$. It follows

$$A\varphi(x) = \int_{\mathbb{R}^n}\sum_{j=1}^m \left(\psi_j(x)\overline{\varphi_j(y)}\right)\varphi(y)dy,$$

i.e. A is an integral operator with kernel $\sum_{j=1}^m \left(\psi_j(x)\overline{\varphi_j(y)}\right) \in L^2(\mathbb{R}^{2n})$. Since the set of finite rank operator is dense in $\mathcal{L}^2(L^2(\mathbb{R}^n))$, we conclude that $\mathcal{A}$ is also surjective. In other words, any Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$ is an integral operator with $L^2(\mathbb{R}^{2n})$-kernel!

In summary, we proved

Theorem 2.1. *An integral operator A_K with Schwartz kernel $K(x,y)$ is a Hilbert-Schmidt integral operator on $L^2(\mathbb{R}^n)$ if and only if $K(x,y) \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Moreover, in this case we have*

$$\|A_K\|_{HS} = \|K\|_{L^2}.$$

¶ Semiclassical PsDOs as Hilbert-Schmidt operators.

As a consequence, we get

Theorem 2.2. *The operator $\mathrm{Op}_\hbar^t(a)$ is a Hilbert-Schmidt operator if and only if $a = a(x,\xi) \in L^2(\mathbb{R}^{2n})$, in which case we have*

$$\|\mathrm{Op}_\hbar^t(a)\|_{HS} = \frac{1}{(2\pi\hbar)^{n/2}} \|a\|^2_{L^2(\mathbb{R}^{2n})}.$$

Proof. We know that the kernel of $\mathrm{Op}_\hbar^t(a)$ is the partial Fourier transform

$$\begin{aligned} k_a^t(x,y) &= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi} a(tx+(1-t)y,\xi)d\xi \\ &= \frac{1}{(2\pi\hbar)^n} [(\mathcal{F}_\hbar)_{\xi\to y-x} a](tx+(1-t)y, y-x). \end{aligned}$$

So we get from the Plancherel's theorem (c.f. Lecture 4)

$$\begin{aligned} \|k_a^t\|_{L^2(\mathbb{R}^{2n})} &= \frac{1}{(2\pi\hbar)^{n/2}} \left[\int_{\mathbb{R}^{2n}} |a(tx+(1-t)x, y-x)|^2 dxdy \right]^{1/2} \\ &= \frac{1}{(2\pi\hbar)^{n/2}} \left[\int_{\mathbb{R}^{2n}} |a(\widetilde{x},\widetilde{y})|^2 d\widetilde{x}d\widetilde{y} \right]^{1/2} \\ &= \frac{1}{(2\pi\hbar)^{n/2}} \|a\|_{L^2(\mathbb{R}^{2n})} \end{aligned}$$

and the conclusion follows. □

In particular, we see that if the order function $m \in L^2(\mathbb{R}^{2n})$, then for any $a \in S(m)$, the semiclassical pseudodifferential operator $\mathrm{Op}_\hbar^t(a)$ is a Hilbert-Schmidt operator whose Hilbert-Schmidt norm is bounded by $O(\hbar^{-n/2})$.

¶ Trace class integral operators.

Now we study trace class integral operators. Unlike Hilbert-Schmidt operators for which we have Theorem 2.1, there is no simple criteria for an integral operator to be a trace class operator. So in what follows we will only give a sufficient condition (on the kernel) so that an integral operator is in trace class.

Since any trace class operator can be written as the composition of two Hilbert-Schmidt operator, we will start with the trace class integral operators which can be written as the composition of two Hilbert-Schmidt integral operators.

Proposition 2.3. *If $K_1, K_2 \in L^2(\mathbb{R}^{2n})$, then $A_{K_1}A_{K_2}$ is a trace class operator with $\|A_{K_1}A_{K_2}\|_{tr} \le \|K_1\|_{L^2(\mathbb{R}^n)} \cdot \|K_2\|_{L^2(\mathbb{R}^n)}$ and*

$$\mathrm{tr}(A_{K_1}A_{K_2}) = \int_{\mathbb{R}^{2n}} K_1(x,y)K_2(y,x)dxdy.$$

Proof. The first half of the proposition is obvious. To calculate the trace we first observe by definition that

$$A^*_{K(x,y)} = A_{\overline{K(y,x)}}.$$

It follows

$$\begin{aligned}
\mathrm{Tr}(A_{K_1}A_{K_2}) &= \sum_j \langle A_{K_1}A_{K_2}\varphi_j, \varphi_j\rangle \\
&= \sum_j \langle A_{K_2}\varphi_j, A^*_{K_1}\varphi_j\rangle \\
&= \sum_j \sum_k \langle A_{K_2}\varphi_j, \varphi_k\rangle \cdot \overline{\langle A^*_{K_1}\varphi_j, \varphi_k\rangle} \\
&= \sum_{j,k} \langle K_2(x,y), \varphi_j(y)\overline{\varphi_k(x)}\rangle_{L^2(\mathbb{R}^{2n})} \cdot \overline{\langle \overline{K_1(y,x)}, \varphi_j(y)\overline{\varphi_k(x)}\rangle}_{L^2(\mathbb{R}^{2n})} \\
&= \int_{\mathbb{R}^{2n}} K_1(y,x)K_2(x,y)dxdy \\
&= \int_{\mathbb{R}^{2n}} K_1(x,y)K_2(y,x)dxdy.
\end{aligned}$$

□

¶ Semiclassical PsDO with Schwartz symbols as trace class operators.

Finally we study the trace property of semiclassical PsDOs. We start with a corollary of Proposition 2.3:[4]

Corollary 2.4. *Suppose there exist $b, c \in L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ such that $\widehat{a}^{KN} = \widehat{b}^{KN} \circ \widehat{c}^{KN}$. Then $\widehat{a}^{KN}$ is a trace class operator with*

$$\mathrm{Tr}(\widehat{a}^{KN}) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} a(x,\xi)dxd\xi.$$

Proof. The Schwartz kernel of $\widehat{b}^{KN}$ is $\frac{1}{(2\pi\hbar)^n}[(\mathcal{F}_\hbar)_{\xi\to y-x}b](x, y-x)$. Thus

$$\mathrm{Tr}(\widehat{b}^{KN} \circ \widehat{c}^{KN}) = \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} [(\mathcal{F}_\hbar)_{\xi\to y-x}b](x,y-x)[(\mathcal{F}_\hbar)_{\xi\to y-x}c](y,x-y)dxdy.$$

[4] The result holds for other quantizations. We stated it via Kohn-Nirenberg quantization only because in this case it is easier to do the computations.

Recall from Lecture 9 that $a = b \star_{KN} c$ has the expression

$$a(x,\xi) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}(\tilde{x}\cdot\tilde{\xi})} b(x,\xi+\tilde{\xi})c(x+\tilde{x},\xi)d\tilde{x}d\tilde{\xi}$$
$$= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}((y-x)\cdot(\eta-\xi))} b(x,\eta)c(y,\xi)dyd\eta$$

which implies

$$\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} a(x,\xi)dxd\xi = \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{4n}} e^{-\frac{i}{\hbar}((y-x)\cdot(\eta-\xi))} b(x,\eta)c(y,\xi)dyd\eta dxd\xi$$
$$= \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}} [(\mathcal{F}_\hbar)_{\xi\to y-x} b](x,y-x)[(\mathcal{F}_\hbar)_{\xi\to y-x} c](y,x-y)dxdy.$$

So we conclude

$$\mathrm{Tr}(\widehat{a}^{KN}) = \mathrm{Tr}(\widehat{b}^{KN}\circ\widehat{c}^{KN}) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} a(x,\xi)dxd\xi.$$

□

Now we can work on the case where the symbols are Schwartz functions. Note that we can easily write a Schwartz function as the product of two L^2-functions. So in view of the function-operator correspondence, we should be able to decompose $Op^t_\hbar(a)$ (where a is a Schwartz function) into the composition of two Hilbert-Schmidt operators, and then use the above theorem to calculate its trace.

Proposition 2.5. *If $a(x,\xi) \in \mathscr{S}(\mathbb{R}^{2n})$, then $\mathrm{Op}^t_\hbar(a)$ is a trace class operator with*

$$\|\mathrm{Op}^t_\hbar(a)\|_{tr} \le C\hbar^{-n}\sum_{|\alpha|\le Md}\hbar^{|\alpha|/2}\|\partial^\alpha a\|_{L^1(\mathbb{R}^{2n})},$$

where M is some constant, and

$$\mathrm{Tr}(\mathrm{Op}^t_\hbar(a)) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} a(x,\xi)dx\ d\xi.$$

Sketch of proof. It is enough to work on the case of Kohn-Nirenberg quantization, since then the general case is a consequence of the special case together with the change of quantization formula. (Work out the detail!) To "decompose" the operator $\widehat{a}^{KN}$, we let $b(x,\xi) = \langle x\rangle^{-n-1}\langle\xi\rangle^{-n-1}$. Then $b \in L^2(\mathbb{R}^{2n})$ and $\widehat{b}^{KN} = \mathcal{F}_\hbar^{-1}\circ \langle x\rangle^{-n-1}\langle\xi\rangle^{-n-1}\circ\mathcal{F}_\hbar$. So if we let $\widehat{c}^{KN} = (\widehat{b}^{KN})^{-1}\circ\widehat{a}^{KN} = (\widehat{\langle x\rangle^{n+1}\langle\xi\rangle^{n+1}})^{KN}\circ\widehat{a}^{KN}$, then the formula for the trace follows immediately since $c = (\langle x\rangle^{n+1}\langle\xi\rangle^{n+1})\star_{KN} a \in \mathscr{S}$.

The trace norm estimate also follows from the decomposition:

$$\|\widehat{a}^{KN}\|_{tr} \le \|\widehat{b}^{KN}\|_{HS}\cdot\|\widehat{c}^{KN}\|_{HS} \le C\hbar^{-n}\sum_{|\alpha|\le Md}\hbar^{|\alpha|/2}\|\partial^\alpha a\|_{L^1(\mathbb{R}^{2n})},$$

where the last step comes from the fact that

$$c(x,\xi) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\tilde{x}\cdot\tilde{\xi}} \langle x\rangle^{n+1} \langle \xi+\tilde{\xi}\rangle^{n+1} a(x+\tilde{x},\xi) d\tilde{x} d\tilde{\xi}.$$

work out the details. □

For the general case we only state the theorem as follows, and leave the proof as an exercise: Again we localize, re-scale, change quantization etc.

Theorem 2.6. *Suppose $a \in S(1)$ satisfies*

$$\sum_{|\alpha|\le Mn} \|\partial^\alpha a\|_{L^1} \le C$$

for all $\hbar \in (0,\hbar_0)$, then the operator $\mathrm{Op}_\hbar^t(a)$ is trace class, with

$$\|\mathrm{Op}_\hbar^t(a)\|_{tr} \le C\hbar^{-d} \sum_{|\alpha|\le Md} \hbar^{|\alpha|/2} \|\partial^\alpha a\|_{L^1(\mathbb{R}^{2n})}$$

for some constant M, and

$$\mathrm{Tr}(\mathrm{Op}_\hbar^t(a)) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2n}} a(x,\xi,\hbar)\ dx\ d\xi.$$