LECTURE 14: L^2-THEORY OF SEMICLASSICAL PsDOs: ELLIPTICITY

Today we study the invertibility of semiclassical pseudodifferential operators: in particular we want to find out conditions so that $\mathrm{Op}_\hbar^t(a)$ is invertible on $L^2(\mathbb{R}^n)$, and the inverse is another semiclassical pseudodifferential operator. Here is an illuminating example: suppose we want to solve a partial differential equation

$$\sum_{|\alpha|\le k} a_\alpha D^\alpha u = f.$$

By applying the Fourier transform, we get

$$(\sum_{|\alpha|\le k} a_\alpha \xi^\alpha)\hat{u} = \hat{f}.$$

So *if* the symbol function $p(x,\xi) = \sum_{|\alpha|\le k} a_\alpha \xi^\alpha$ is invertible, then we get

$$\hat{u} = p(x,\xi)^{-1}\hat{f}$$

and thus at least formally,

$$u = \mathcal{F}^{-1}(p(x,\xi)^{-1}\mathcal{F}f) = (\widehat{p(x,\xi)^{-1}})^{KN}(f).$$

In fact this argument is one of the original motivations of introducing pseudodifferential operators in mathematics, and it fits into our general philosophy perfectly: the invertibility of a (semiclassical) pseudodifferential operator is closely related to the invertibility of its symbol function! Of course we should be a little bit more careful: not only we want $p(x,\xi)$ to be invertible, but also we want the inverse $p(x,\xi)$ to be in some nice symbol class.

1. Ellipticity

¶ Definition of ellipticity.

Now we study the invertibility of $\widehat{a}^W$. By the above argument, we see that for $\widehat{a}^W$ to be invertible, we want a to be invertible as a function. Moreover, suppose $a \in S(m)$, where $m = m(x,\xi)$ be an order function on $\mathbb{R}^{2n}$. Then we want the inverse $1/a$ to be in some nice symbol class. For example, if $a \in S(1)$, then it is natural to require $1/a \in S(1)$ (so that the quantization of the inverse is still a bounded linear operator). This amounts to require the function $|1/a|$ to be bounded above, or equivalently, require $|a|$ to be bounded below:

$$|a(x,\xi,\hbar)| \ge C > 0, \quad \forall (x,\xi) \in \mathbb{R}^{2n}, \hbar \in (0,1]^1.$$

Similarly for $a \in S(m)$, the natural symbol class for $1/a$ is $1/m$, and thus we want the function $|1/a|$ to be bounded above by Cm. As a result, it is natural to require that $|a|$ is bounded blow by a constant multiple of the order function m. It turns out that this condition is also sufficient to guarantee that the inverse $1/a$ is in $S(1/m)$:

Lemma 1.1. *Suppose there exists $C > 0$ such that $|a(x,\xi,\hbar)| \ge Cm$ holds for all $(x,\xi) \in \mathbb{R}^{2n}$ and all $\hbar \in (0,1]$, then $1/a \in S(1/m)$.*

Proof. A tedious computation shows that $\partial^\alpha(1/a)$ has the form

$$\partial^\alpha(\frac{1}{a}) = \frac{1}{a} \sum_{\beta_1+\cdots+\beta_k=\alpha, |\beta_j|\ge 1} C_{\beta_1\cdots\beta_k} \prod_j (\frac{1}{a}\partial^{\beta_j} a).$$

The conclusion follows. □

So we define

Definition 1.2. We say a symbol $a \in S(m)$ is (semiclassically) *elliptic in $S(m)$* if there exists a constant $C_0 > 0$ (independent of $\hbar$) such that

$$|a(x,\xi)| \ge C_0 m \tag{1}$$

on $\mathbb{R}^{2n}$.

So by Lemma 1.1, if $a \in S(m)$ is elliptic, then $\frac{1}{a} \in S(\frac{1}{m})$. Note that by the formula in the proof of Lemma 1.1, we can say more: if $a \in S_\delta(m)$ is elliptic in $S(m)$, then $\frac{1}{a} \in S_\delta(\frac{1}{m})$.

Throughout this lecture we always assume $0 \le \delta < 1/2$.

¶ Constructing parametrix.

First we prove that for an elliptic symbol a, the operator $\widehat{a}^W$ is "almost invertible":

Theorem 1.3. *Suppose there exists $0 \le \delta < \frac{1}{2}$ and an order function m so that $a \in S_\delta(m)$ is elliptic in $S(m)$. Then there exists $b, c \in S_\delta(\frac{1}{m})$ so that*[2]

$$\begin{aligned} \widehat{a}^W \circ \widehat{b}^W &= \mathrm{Id} + \widehat{r_1}^W, \\ \widehat{c}^W \circ \widehat{a}^W &= \mathrm{Id} + \widehat{r_2}^W \end{aligned} \tag{2}$$

for some $r_1, r_2 = O(\hbar^\infty)$ in $S(1)$. Moreover, for $m = 1$ we can take $b = c$.

[1] In previous lectures we wrote $\hbar \in (0,\hbar_0)$ and we used the special case $\hbar = 1$. This is not a serious problem because we can always do rescaling.

[2] Both equations are understood to be hold on dense subsets of $L^2(\mathbb{R}^n)$ which contain $\mathscr{S}$.

Proof. The idea is to approximate the inverse step by step: in each step we use a "smaller" adjustment (of order $O(\hbar^k)$) to get a better approximation (with error term of order $O(\hbar^k+1)$), and then we add all adjustments via Borel's lemma to get the $O(\hbar^\infty)$-approximated inverse that we want.

So of course the first step is to start with $\tilde{b}_1 = 1/a$, then $\tilde{b}_1 \in S_\delta(1/m)$. According to the composition formula (see the remark at the end of Lecture 10),

$$a \star \tilde{b}_1 = 1 - \hbar^{1-2\delta}\tilde{r}_1 \tag{3}$$

for some $\tilde{r}_1 \in S_\delta(1)$. This is of course only the first approximation. To get better approximation, we may try to find $\tilde{b}_2$ so that

$$a \star (\tilde{b}_1 + \tilde{b}_2) = 1 - \hbar^{2(1-2\delta)}\tilde{r}_2 \tag{4}$$

for some $\tilde{r}_2 \in S_\delta(1)$. Plugging (3) into (4), we see that we need to find $\tilde{b}_2$ so that

$$a \star \tilde{b}_2 = \hbar^{1-2\delta}(\tilde{r}_1 - \hbar^{1-2\delta}\tilde{r}_2).$$

Comparing this "target equation" with the equation (3), we can easily find a candidate for $\tilde{b}_2$: if we take $\tilde{b}_2 = \hbar^{1-2\delta}\tilde{b}_1 \star \tilde{r}_1$, then (4) is fulfilled; more over the candidate is *perfect* because

- we have $\tilde{b}_2 = \hbar^{1-2\delta}\tilde{b}_1 \star \tilde{r}_1 \in \hbar^{1-2\delta}S_\delta(1/m)$, which is "smaller" than $\tilde{b}_1$.
- we have $\tilde{r}_2 = \tilde{r}_1 \star \tilde{r}_1 \in S_\delta(1)$, so the remainder $\hbar^{2(1-2\delta)}\tilde{r}_2 \in O(\hbar^{2(1-2\delta)})$ is even smaller.

Now we repeat the process. It is easy to see that if we take

$$\tilde{b}_{k+1} = \hbar^{k(1-2\delta)}\tilde{b}_1 \star \tilde{r}_1 \star \cdots \star \tilde{r}_1,$$

then $\tilde{b}_{k+1} \in \hbar^{k(1-2\delta)}S_\delta(1/m)$ and

$$a \star (\tilde{b}_1 + \tilde{b}_2 + \cdots + b_{k+1}) = 1 - \hbar^{(k+1)(1-2\delta)}\tilde{r}_{k+1},$$

where $r_{k+1} = \tilde{r}_1 \star \cdots \star r_1 \in S_\delta(1)$. Now we apply Borel's Lemma (c.f. Lecture 10) to get $b \in S(\frac{1}{m})$ which is the asymptotic sum

$$b \sim \tilde{b}_1 + \tilde{b}_2 + b_3 + \cdots .$$

Then

$$a \star b = 1 + O(\hbar^\infty).$$

It follows that

$$\widehat{a}^W \circ \widehat{b}^W = \mathrm{Id} + \widehat{r_1}^W$$

for some $r_1 = O(\hbar^\infty)$ in $S(1)$. This gives the first identity.

To prove the second identity, we first repeat the same argument above to find $c \in S(\frac{1}{m})$ so that

$$\widehat{c}^W \circ \widehat{a}^W = \mathrm{Id} + \widehat{r_2'}^W$$

for some $r_2' = O(\hbar^\infty)$ in $S(1)$.

Finally we assume $m = 1$, in which case all operators involved are bounded linear operators and thus are defined on the whole $L^2(\mathbb{R}^n)$. Since

$$(\mathrm{Id} + \widehat{r_2'}^W) \circ \widehat{b}^W = \widehat{c}^W \circ \widehat{a}^W \circ \widehat{b}^W = \widehat{c}^W \circ (\mathrm{Id} + \widehat{r_1}^W)$$

we see

$$\widehat{b}^W = \widehat{c}^W + \widehat{c}^W \circ \widehat{r_1}^W - \widehat{r_2'}^W \circ \widehat{b}^W = \widehat{c}^W + \widehat{r}^W$$

for some $r = O(\hbar^\infty)$ in $S(\frac{1}{m})$. It follows

$$\widehat{b}^W \circ \widehat{a}^W = \widehat{c}^W \circ \widehat{a}^W + \widehat{r}^W \circ \widehat{a}^W = \mathrm{Id} + \widehat{r_2}^W$$

for some $r_2 = O(\hbar^\infty)$ in $S(1)$. □

The operator $\widehat{b}^W$ is called a *parametrix* of $\widehat{a}^W$.

¶ The inverse of elliptic $\hbar$-PsDO's.

Since $r_1 = O(\hbar^\infty)$ in $S(1)$, by Calderon-Vaillancourt Theorem, $\widehat{r_1}^W$ is a bounded linear operator on $L^2(\mathbb{R}^n)$, and moreover, for $\hbar_0$ small enough,

$$\|\widehat{r_1}^W\|_{\mathcal{L}(L^2(\mathbb{R}^n))} < 1/2, \qquad \forall \hbar \in (0, \hbar_0).$$

It follows that for $\hbar \in (0, \hbar_0)$, the operator $(\mathrm{Id} + \widehat{r_1}^W)^{-1}$ exists and is the bounded linear operator given by the Neumann series

$$(\mathrm{Id} + \widehat{r_1}^W)^{-1} = \sum_{k \geq 0} (-1)^k (\widehat{r_1}^W)^k.$$

Moreover, $\|(\mathrm{Id} + \widehat{r_1}^W)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq 2$.

It thus follows from the first equation in (2) that

$$\widehat{a}^W \circ (\widehat{b}^W \circ (\mathrm{Id} + \widehat{r_1}^W)^{-1}) = \mathrm{Id}.$$

Similarly we have

$$((\mathrm{Id} + \widehat{r_2}^W)^{-1} \circ \widehat{b}^W) \circ \widehat{a}^W = \mathrm{Id}.$$

Note that if $a \in S(1)$, then $b \in S(1)$. So all operators appeared in the previous two formulaes are bounded linear operators (thus well-defined) on $L^2(\mathbb{R}^n)$, and in this case get

Corollary 1.4. *Suppose $a \in S_\delta(1)$ is elliptic in $S(1)$. Then $\widehat{a}^W$ is invertible with*

$$(\widehat{a}^W)^{-1} = \widehat{b}^W \circ (\mathrm{Id} + \widehat{r_1}^W)^{-1} = (\mathrm{Id} + \widehat{r_2}^W)^{-1} \circ \widehat{b}^W.$$

Remark. In the proof, instead of using r_1, we may simply use the error of the first approximation, namely $1 - a \star (1/a) = \hbar^{1-2\delta}\tilde{r}_1 = O(\hbar^{1-2\delta}) \in S(1)$.

Another simple consequence from the boundedness of $(\mathrm{Id} + \widehat{r_2}^W)^{-1}$ is the following classical-quantum correspondence phenomena: since the absolute value of an elliptic symbol has a positive lower bound (w.r.t. m), the operator norm of its quantized operator also has a positive lower bound.

Corollary 1.5. *Suppose $a \in S_\delta(m)$ is elliptic in $S(m)$. Suppose $m \geq 1$. Then there exists $\hbar_0 > 0$ and $C > 0$ such that for all $\hbar \in (0, \hbar_0)$,*

$$\|\widehat{a}^W\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \geq C.$$

Proof. Since $m \geq 1$, $c \in S(1/m) \subset S(1)$. So $\widehat{c}^W$ is bounded. It follows

$$\|u\|_{L^2(\mathbb{R}^n)} = \|((\mathrm{Id} + \widehat{r_2}^W)^{-1} \circ \widehat{c}^W) \circ \widehat{a}^W u\|_{L^2(\mathbb{R}^n)} \leq C\|\widehat{a}^W u\|_{L^2(\mathbb{R}^n)}.$$

□

¶ The adjoint action.

The invertibility of $\widehat{a}^W$ (for elliptic $a \in S(1)$) on the space $L^2(\mathbb{R}^n)$ (as a bounded linear operator) is not that satisfying, since we don't know whether the inverse of $\widehat{a}^W$ is still a semiclassical pseudodifferential operator. What we need is *a criterion for a bounded linear operator A on $L^2(\mathbb{R}^n)$ to be the quantization of a bounded symbol $a \in S(1)$.*

For this purpose we introduce a notation, the *adjoint action* between operators: For operators A and B we denote

$$\mathrm{ad}_A B = [A, B] = AB - BA. \tag{5}$$

Example. In PSet 2-6 we used the relation

$$[A, BC] = [A, B]C + B[A, C].$$

Using this new notation, we can rewrite it as

$$\mathrm{ad}_A(BC) = (\mathrm{ad}_A B)C + B(\mathrm{ad}_A C).$$

If B is invertible and we take $C = B^{-1}$ in the above formula, we get

$$\mathrm{ad}_A(B^{-1}) = -B^{-1}(\mathrm{ad}_A B)B^{-1}. \tag{6}$$

Example. In Lecture 9 we have seen (whose proof was left as an exercise in PSet 2) that for a linear symbol $l(x, \xi) = c \cdot x + d \cdot \xi$, the quantization condition is exact, namely

$$[\widehat{l}^W, \widehat{a}^W] = \frac{\hbar}{i}\widehat{\{l, a\}}^W.$$

Using the adjoint action, we can rewrite this as

$$\mathrm{ad}_{\widehat{l}^W}\widehat{a}^W = \frac{\hbar}{i}\widehat{\{l, a\}}^W. \tag{7}$$

¶ Semi-classical Beals's theorem.

Now we try to search for the criterion. Suppose $A = \widehat{a}^W$ for some $a \in S(1)$. Then for any linear function $l = l(x,\xi) = c \cdot x + d \cdot \xi$, the Poisson bracket

$$\{l, a\} = \sum (d_j \partial_{x_j} a + c_j \partial_{\xi_j} a) \in S(1).$$

As a consequence, for any N and any linear functions $l_1, \cdots, l_N$ on $\mathbb{R}^{2n}$, we have

$$\|\mathrm{ad}_{\widehat{l_1}^W} \circ \cdots \circ \mathrm{ad}_{\widehat{l_N}^W} A\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = O(\hbar^N).$$

It turns out that this condition is not only necessary but also sufficient condition for an operator A to be the quantization of a bounded symbol $a \in S(1)$:

Theorem 1.6 (Semiclassical Beals's theorem)**.** *Let $A : \mathscr{S} \to \mathscr{S}'$ be a continuous linear operator. Then $A = \widehat{a}^W$ for a symbol $a \in S(1)$ if and only if for all $N = 0, 1, 2, \cdots$ and all linear functions $l_1, \cdots, l_N$ on $\mathbb{R}^{2n}$, we have*

$$\|\mathrm{ad}_{\widehat{l_1}^W} \circ \cdots \circ \mathrm{ad}_{\widehat{l_N}^W} A\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = O(\hbar^N). \tag{8}$$

We postpone the proof to the end of this lecture.

¶ The inverse of an elliptic $\hbar$-PsDO as an PsDO.

Now we apply semiclassical Beals' theorem to study the left and right inverses of $\widehat{a}^W$, where $a \in S_\delta(m)$ is elliptic. It is enough to consider the following question: given a symbol r which is $O(\hbar^\infty)$ in $S(1)$. Assume $\|\widehat{r}^W\|_{\mathcal{L}(L^2(\mathbb{R}^n))} < 1/2$. Is the inverse

$$(\mathrm{Id} + \widehat{r}^W)^{-1} = \sum_{k \geq 0} (-1)^k (\widehat{r}^W)^k$$

a semiclassical pseudodifferential operator? The answer is yes. In fact, since $r \in \hbar S(1)$ (so we only need to use $r = r_k$ in the proof of Theorem 1.3 for some $k \geq 1/(1-2\delta)$), Beals's theorem implies

$$\|\mathrm{ad}_{\widehat{l_1}^W} \circ \cdots \circ \mathrm{ad}_{\widehat{l_N}^W} (\mathrm{Id} + \widehat{r}^W)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = O(\hbar^N)$$

for any linear functions $l_1, \cdots, l_N$ on $\mathbb{R}^{2n}$. Now we play a magic: we apply the formula (6) recursively to conclude

$$\|\mathrm{ad}_{\widehat{l_1}^W} \circ \cdots \circ \mathrm{ad}_{\widehat{l_N}^W} (\mathrm{Id} + \widehat{r}^W)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = O(\hbar^N).$$

So according to the semiclassical Beals's theorem, the operator $(\mathrm{Id} + \hbar\widehat{r}^W)^{-1}$ is a semiclassical pseudodifferential operator, i.e. $\exists c \in S(1)$ such that $(\mathrm{Id} + \hbar\widehat{r}^W)^{-1} = \widehat{c}^W$.

In conclusion, we proved

Theorem 1.7. *Suppose $a \in S_\delta(1)$ is elliptic elliptic in $S(1)$, then as a bounded linear operator on $L^2(\mathbb{R}^n)$, $\widehat{a}^W$ is invertible. Moreover, there exists $b \in S_\delta(1)$ so that*

$$(\widehat{a}^W)^{-1} = \widehat{b}^W.$$

Similarly if $a \in S_\delta(m)$ is elliptic in $S(m)$, and if $m \ge 1$, then by repeating the arguments above, we can prove

- $\widehat{a}^W$ has a left/right inverse which are bounded linear operators.
- the left/right inverses of $\widehat{a}^W$ are semiclassical pseudodifferential operators with symbol in $S_\delta(1/m)$.

If we denote the left/right inverse of $\widehat{a}^W$ by $\widehat{b}^W$ and $\widehat{c}^W$, then at least on $\mathscr{S}$ (we use the fact that for a in any symbol class $S(m)$, $\widehat{a}^W$ maps $\mathscr{S}$ to $\mathscr{S}$), we have

$$\widehat{b}^W = \widehat{b}^W \circ \widehat{a}^W \widehat{c}^W = \widehat{c}^W.$$

But since both $\widehat{b}^W$ and $\widehat{c}^W$ are continuous linear functionals on $L^2(\mathbb{R}^n)$, and since $\mathscr{S}$ is dense in $L^2(\mathbb{R}^n)$, we conclude that $\widehat{b}^W = \widehat{c}^W$ on $L^2(\mathbb{R}^n)$. In other words,

Corollary 1.8. *If $a \in S_\delta(m)$ is elliptic in $S(m)$, and if $m \ge 1$, then there exists $b \in S_\delta(1/m)$ so that $\widehat{a}^W \circ \widehat{b}^W = \mathrm{Id}$ and $\widehat{b}^W \circ \widehat{a}^W = \mathrm{Id}$.*

We will call this $\widehat{b}^W$ the *inverse* of $\widehat{a}^W$.

2. Proof of the semiclassical Beals's theorem

¶ The proof of the semiclassical Beals's theorem.

Proof. We have seen that if $a \in S(1)$, then $A = \widehat{a}^W$ satisfies (8).

Conversely, suppose the estimate (8) holds. We first assume $\hbar = 1$. By the Schwartz kernel theorem that we mentioned in Lecture 6,

$$Au(x) = \int_{\mathbb{R}^n} K_A(x,y)u(y)dy$$

for some $K_A \in \mathscr{S}'(\mathbb{R}^{2n})$. According to the Fourier inversion formula, we have

$$K_A(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y-w)\cdot\xi} K_A(\frac{x+y}{2}+\frac{w}{2}, \frac{x+y}{2}-\frac{w}{2})dwd\xi.$$

So if we set

$$a(x,\xi) = \int_{\mathbb{R}^n} e^{-iw\cdot\xi} K_A(x+\frac{w}{2}, x-\frac{w}{2})dw,$$

then

$$K_A(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(\frac{x+y}{2},\xi)d\xi,$$

i.e. $A = \widehat{a}^W|_{\hbar=1}$. It remains to prove $a \in S(1)$. To do so, we apply our hypothesis to functions $l_j = x_j$ and $l_j = \xi_j$ and use (7) to conclude that

$$\|(\widehat{\partial^\alpha a}^W)_{\hbar=1}\|_{\mathcal{L}(L^2)} \le C_\alpha$$

for all multi-indices α. It remains to control the L^∞-norm of $\partial^\alpha a$ from the operator norm of the operators $\widehat{\partial^\rho a}^W$. For this purpose we need the following "reverse" to the Calderon-Vailancourt theorem:

(Recall: The Calderon-Vailancout tells us that the operator norm of $\widehat{a}^W$ is bounded by the L^∞-norm of $\partial^\alpha a$ for finitely many α. So the following theorem is a reverse:)

Theorem 2.1. *Let $a \in S'(\mathbb{R}^{2n})$ and assume that for all multi-indices γ with $|\gamma| \leq 2n+1$, the operator $\widehat{\partial^\gamma a}^W \in \mathcal{L}(L^2(\mathbb{R}^n))$. Then $a \in L^\infty(\mathbb{R}^n)$ and*

$$\|a\|_{L^\infty} \leq C \sum_{|\gamma| \leq Mn} \hbar^{|\gamma|/2} \|\widehat{\partial^\gamma a}^W\|_{\mathcal{L}(L^2(\mathbb{R}^n))}. \tag{9}$$

So by Theorem 2.1,

$$\sup_{\mathbb{R}^{2n}} |\partial^\alpha a| \leq C_\alpha.$$

In other words, $a \in S(1)$.

Finally one can use the re-scaling trick to convert the general case to the $\hbar = 1$ case. ☐

¶ An "reverse" to the Calderon-Vaillancourt Theorem.

It remains to prove Theorem 2.1.

Proof. We will only prove the theorem for $\hbar = 1$ and for the Kohn-Nirenberg quantization. The general case follows from re-scaling trick and the change of quantization formula.

Again we prove a local estimate and then globalize it, this time since we are estimating the L^∞-norm, we don't even need to choose a partition of unity: to globalize the estimate it is enough to choose a cut-off function that is positive locally, prove a local L^∞ estimate, then use translation to get the demanded global estimate.

So we let $\varphi = \varphi(x)$ and $\hat{\psi} = \hat{\psi}(\xi)$ be functions in $\mathscr{S}$ which equals 1 near 0. Put

$$\chi(x,\xi) = \overline{\varphi}(x)\widehat{\psi}(\xi)e^{ix\cdot\xi}.$$

Then $\chi \in \mathscr{S}(\mathbb{R}^{2n})$, and $|\chi(x,\xi)| = 1$ near $(0,0)$.

The local estimate that we need is

Lemma 2.2 (Local L^∞-estimate)**.** *Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$ and assume that*

$$\widehat{\partial^\gamma a}^{KN} \in \mathcal{L}(L^2(\mathbb{R}^n)), \qquad \forall \gamma \text{ with } |\gamma| \leq 2n+1. \tag{10}$$

Then

$$\|\chi a\|_{L^\infty} \leq C \sum_{|\gamma| \leq 2n+1} \|\widehat{\partial^\gamma a}^{KN}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}.$$

We postpone the proof to the end.

We denote $a_{y,\eta}(x,\xi) = a(x+y,\xi+\eta)$ be the translation of a by (y,η). Since $|\chi(x,\xi)| = 1$ in a neighborhood of $(0,0)$, we have

$$\|a\|_{L^\infty} \le \sup_{(y,\eta)\in\mathbb{R}^{2n}} \|\chi a_{y,\eta}\|_{L^\infty} \le \sup C_\chi \sum_{|\gamma|\le 2n+1} \|\widehat{\partial^\gamma a_{y,\eta}}^{KN}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}.$$

On the other hand, one can check that if we set $U: L^2 \to L^2$ to be the operator

$$Uv(x) = e^{-i\eta\cdot x} v(x+y),$$

then U is unitary and

$$\widehat{a_{y,\eta}}^{KN} = U\widehat{a}^{KN} U^*.$$

It follows

$$\|\widehat{\partial^\gamma a_{y,\eta}}^{KN}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = \|\widehat{\partial^\gamma a}^{KN}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}.$$

This completes the proof. □

¶ Proof of the local L^∞ estimate.

Finally finally, it remains to prove the L^∞ estimate, namely, Lemma 2.2.

We first observe that

$$\|\chi a\|_{L^\infty(\mathbb{R}^{2n})} \le \frac{1}{(2\pi)^{2n}} \|\mathcal{F}(\chi a)\|_{L^1(\mathbb{R}^{2n})} \le C\|\langle (y,\eta)\rangle^{2n+1}\mathcal{F}(\chi a)\|_{L^\infty(\mathbb{R}^{2n})}.$$

Now for any (y,η), we have

$$\begin{aligned}\left|y^\alpha \eta^\beta \mathcal{F}_{(x,\xi)\to(y,\eta)}(\chi a)(y,\eta)\right| &= \left|\mathcal{F}_{(x,\xi)\to(y,\eta)} D_x^\alpha D_\xi^\beta(\chi a)\right| \\ &\le C \sum_{|\rho|,|\gamma|\le|\alpha+\beta|} \left|\mathcal{F}_{(x,\xi)\to(y,\eta)} D_{x,\xi}^\rho a D_{x,\xi}^\gamma \chi\right|\end{aligned}$$

Since $\chi(x,\xi) = \overline{\varphi}(x)\widehat{\psi}(\xi)e^{ix\cdot\xi}$ we have

$$\begin{aligned}&\left|\mathcal{F}_{(x,\xi)\to(y,\eta)}(D_{x,\xi}^\rho a D_{x,\xi}^\gamma \chi)\right| \\ &= \left|\int_{\mathbb{R}^{2n}} e^{-i(x,\xi)\cdot(y,\eta)} (D_{x,\xi}^\rho a) D_x^{\gamma_1}\overline{\varphi}(x) D_\xi^{\gamma_2}\widehat{\psi}(\xi) x^{\gamma_3}\xi^{\gamma_4} e^{ix\cdot\xi} dx d\xi\right| \\ &= \left|\int_{\mathbb{R}^{2n}} (D_{x,\xi}^\rho a)\left(x^{\gamma_3} D_x^{\gamma_1}\overline{\varphi}(x) e^{-ix\cdot y}\right) \mathcal{F}_{z\to\xi}(D_z^{\gamma_4}(z+\eta)^{\gamma_2}\psi(z+\eta)) e^{ix\cdot\xi} dx d\xi\right| \\ &= \left|\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{2n}} e^{i(x-z)\cdot\xi}(D_{x,\xi}^\rho a)(D_z^{\gamma_4}(z+\eta)^{\gamma_2}\psi(z+\eta)) dz d\xi\right)\left(x^{\gamma_3} D_x^{\gamma_1}\overline{\varphi}(x) e^{-ix\cdot y}\right) dx\right| \\ &= \left|\left\langle \widehat{(D_{x,\xi}^\rho a)}^{KN} (D_z^{\gamma_4}(z+\eta)^{\gamma_2}\psi(z+\eta)),\ \overline{x^{\gamma_3} D_x^{\gamma_1}\overline{\varphi}(x) e^{-ix\cdot y}}\right\rangle_{L^2(\mathbb{R}^n)}\right| \\ &\le \left\|\widehat{(D_{x,\xi}^\rho a)}^{KN}\right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \cdot |D_z^{\gamma_4}(z+\eta)^{\gamma_2}\psi(z+\eta)\|_{L^2(\mathbb{R}^n)} \cdot \left\|x^{\gamma_3} D_x^{\gamma_1}\overline{\varphi}(x) e^{-ix\cdot y}\right\|_{L^2(\mathbb{R}^n)}\end{aligned}$$

Since $\varphi, \psi \in \mathscr{S}$, we get

$$\|\chi a\|_{L^\infty} \leq C \sum_{|\gamma| \leq 2n+1} \|\widehat{\partial^\gamma a}^{KN}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}.$$