LECTURE 16: GENERALIZED SOBOLEV SPACES

1. Generalized Sobolev spaces

For most of the previous five lectures, we are studying $\widehat{a}^W$ for $a \in S(1)$, since in this case $\widehat{a}^W$ is a bounded linear operator on $L^2(\mathbb{R}^n)$. A natural question is: for more general m, what can be said for $\widehat{a}^W$ with $a \in S(m)$? In particular for those m which diverges to $+\infty$ as $z \to \infty$, what can we say about $\widehat{a}^W$ with $a \in S(m)$?

¶ The Sobolev space.

Let's start with an example. Consider

$$a(x, \xi) = |\xi|^2.$$

Then we have

$$\widehat{a}^W = -\hbar^2 \Delta,$$

which is of course one of the most important operators in geometry and analysis. We may take

$$m(x, \xi) = \langle \xi \rangle^2.$$

Then it is easy to see $a \in S(m)$. Since $a \notin S(1)$, the operator Δ is unbounded (and thus is only densely defined) on $L^2(\mathbb{R}^n)$. However, it is a standard fact form PDE that Δ admits a natural domain: the Sobolev space

$$H^2(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \mid \Delta u \in L^2(\mathbb{R}^n)\},$$

which is a Hilbert space with the Sobolev norm

$$\|u\|_{H^2(\mathbb{R}^n)} = \sum_{|\alpha| \le 2} \|D^\alpha u\|_{L^2(\mathbb{R}^n)}.$$

Here is another way to think of the Sobolev space $H^2(\mathbb{R}^n)$:

$$(I - \hbar^2 \Delta) H^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)$$

By definition we have $(I - \hbar^2 \Delta) H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. To prove the reverse inclusion, for any $f \in L^2(\mathbb{R}^n)$ we need to solve the PDE

$$-\hbar^2 \Delta u + u = f.$$

We have solved such equations via Fourier transform at the beginning of Lecture 14:

$$u(x) = \mathcal{F}_\hbar^{-1}\left(\frac{1}{1 + |\xi|^2} \mathcal{F}_\hbar(f)\right).$$

It remains to check $u \in L^2(\mathbb{R}^n)$ and $\Delta u \in L^2(\mathbb{R}^n)$, both of which are consequences of the fact $\mathcal{F}_\hbar$ is an isomorphism on $L^2(\mathbb{R}^n)$.

Note that $I - \hbar^2\Delta$ is invertible since its symbol $1+|\xi|^2$ is elliptic in $S(m)$, where $m(x,\xi) = \langle\xi\rangle^2$. So we may rewrite the above equation as

$$H^2(\mathbb{R}^n) = (I - \hbar^2\Delta)^{-1} L^2(\mathbb{R}^n).$$

Recall: although $I - \hbar^2\Delta$ is only densely defined, the inverse $(I-\hbar^2\Delta)^{-1}$ is a globally defined compact operator on $L^2(\mathbb{R}^n)$.

The Sobolev norm can also be defined via the operator Δ. More precisely, one can prove that the Sobolev norm alluded to above is equivalent to the Sobolev norm

$$\|u\|_{H^2_\hbar(\mathbb{R}^n)} := \|(I - \hbar^2\Delta)u\|_{L^2(\mathbb{R}^n)}.$$

In other words, the Sobolev space is not only a space defined *for* the Laplace operator Δ, but also a space defined *via* the operator Δ.

¶ The generalized Sobolev spaces.

Observe that the most important thing in the above discussion is that the symbol $1 + \|\xi\|^2$ is elliptic in $S(m)$, where $m(x,\xi) = \langle\xi\rangle^2 \geq 1$. Inspired by this observation, we may define, for any order function $m \geq 1$ and any elliptic symbol $g \in S(m)$, the following *generalized Sobolev norm*

$$\|u\|_{H_\hbar(m,g)} := \|\widehat{g}^W u\|_{L^2}.$$

We know that this is well-defined at least for all $u \in \mathscr{S}$. Let's first investigate the dependence of this norm with the elliptic symbol g. It turns out that the norm is "almost" independent of g and thus is essentially an intrinsic property of the order function m:

Lemma 1.1. *Suppose $m \geq 1$ and g, g' are two elliptic symbols in $S(m)$. Then the generalized Sobolev norms defined via g and g' are equivalent: there exists $\hbar_0 > 0$ and $C > 0$ such that for all $\hbar \in (0, \hbar_0)$,*

$$\frac{1}{C}\|u\|_{H_\hbar(m,g)} \leq \|u\|_{H_\hbar(m,g')} \leq C\|u\|_{H_\hbar(m,g)}, \quad \forall u \in \mathscr{S}.$$

Proof. Since g is elliptic in $S(m)$ and $m \geq 1$, there exists $\hbar_0 > 0$ and $h \in S(1/m)$ such that $(\widehat{g}^W)^{-1} = \widehat{h}^W$ for all $\hbar \in (0, \hbar_0)$. It follows that $g' \star h \in S(1)$ and thus there exists $C > 0$ such that

$$\|\widehat{g'}^W \circ \widehat{h}^W\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C$$

(uniform for all $\hbar \in (0, \hbar_0)$). It follows

$$\|u\|_{H_\hbar(m,g')} = \|\widehat{g'}^W \circ \widehat{h}^W \circ \widehat{g}^W u\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H_\hbar(m,g)}.$$

The other half can be proved by exchanging g and g' above. □

As a consequence, in the definition of the generalized Sobolev norm, we may erase g and simply denote it by $\|\cdot\|_{H_\hbar(m)}$:

Definition 1.2. We will define the *generalized Sobolev norm* associated to m to be

$$\|u\|_{H_\hbar(m)} := \|\widehat{g}^W u\|_{L^2}.$$

We will denote the completion of $\mathscr{S}$ under the norm $\|\cdot\|_{H_\hbar(m)}$ by $H_\hbar(m)$, and call it the *generalized Sobolev space* associated to m.

Note that in the proof of the inequality $\|u\|_{H_\hbar(m,g')} \le C\|u\|_{H_\hbar(m,g)}$ above, we only used the fact that $g' \star h$ is a bounded symbol. In particular, we conclude: if g is elliptic in $S(m)$, g' is elliptic in $S(m')$, and if $m' \le m$, then there exists $C > 0$ such that $\|u\|_{H_\hbar(m')} \le C\|u\|_{H_\hbar(m)}$. In other words, we have:

Corollary 1.3. *If $m' \le m$, then $H_\hbar(m) \subset H_\hbar(m')$.*

¶ DETOUR: Choice of order function.

One may ask: is there any canonical way to choose an elliptic symbol g in $S(m)$? For example, in the standard Sobolev space case, we used the elliptic symbol $1+|\xi|^2$, which is in fact the same as $m = \langle\xi\rangle^2$. Note that by definition, if $m \in S(m)$, then m is automatically elliptic in $S(m)$. Recall that

- a continuous function m on $\mathbb{R}^d$ is an order function if $m(z) \le C\langle z-w\rangle^N m(w)$.
- $S(m)$ contains those smooth functions all of whose derivatives are bounded by the function m.

So in general it is not always true that $m \in S(m)$: m could be non-smooth, or smooth but quite "oscillating" so that its derivatives are not nicely bounded. However, our experience from analysis tells us that there is a big chance that these bad behaviors could be eliminated by using convolution:

Lemma 1.4. *For any order function m, there exists an order function $\widetilde{m}$ such that*

(1) $S(\widetilde{m}) = S(m)$.
(2) $\widetilde{m} \in S(\widetilde{m})$

Proof. Take a cut-off function $\eta \in C_0^\infty(\mathbb{R}^d)$ with $\eta \ge 0$ and $\int \eta dz = 1$. Let

$$\widetilde{m}(z) = m * \eta(z) = \int m(z-w)\eta(w)dw$$

be the convolution of m and η. According to the definition of an order function,

$$C^{-1}\langle w\rangle^{-N} \le \frac{m(z-w)}{m(z)} \le C\langle w\rangle^N.$$

It follows

$$C^{-1}m \le \widetilde{m} \le Cm,$$

which implies $S(m) = S(\widetilde{m})$.

Moreover, for any multi-index α, by commutativity of convolution we have

$$|\partial^\alpha \widetilde{m}| = |m * \partial^\alpha \eta| \le C_\alpha m,$$

so $\widetilde{m} \in S(m) = S(\widetilde{m})$. $\square$

In what follows we will always assume $m \in S(m)$, so that in the definition of $H_\hbar(m)$, we can simply take $g = m$.

We remark that as a direct consequence of $m \in S(m)$ and the formula for $\partial^\alpha \frac{1}{a}$ that we used a couple times, we have $m^{-1} \in S(m^{-1})$. More generally $m^t \in S(m^t)$ for any $t \in \mathbb{R}$. (Reason: Let $|\alpha| \geq 1$. Without loss of generality, we may assume $\alpha_1 \geq 1$ and denote $\tilde{\alpha} = (\alpha_1 - 1, \alpha_2, \cdots, \alpha_d)$. Then

$$\partial^\alpha \log a = \partial^{\tilde{\alpha}}(a^{-1}\partial_1 a) = \sum_{\beta+\gamma=\tilde{\alpha}} \binom{\tilde{\alpha}}{\beta} \partial^\beta(a^{-1})\partial^\gamma(\partial_1 a). \tag{1}$$

So if $m \in S(m)$, then $\partial^\alpha \log m$ is bounded for any $|\alpha| \geq 1$. Since $m^t = e^{t \log m}$, we immediately get $m^t \in S(m^t)$.) As a consequence, we see $\langle \xi \rangle^t \in S(\langle \xi \rangle^t)$ for any t.

¶ The generalized Sobolev spaces: examples.

Example. If $m = 1$, then $H_\hbar(m) = L^2(\mathbb{R}^n)$.

Example. More generally let $m = m(x)$ be a smooth function that depends only on x and suppose $m \in S(m)$. Then $\widehat{m}^W$ is the "multiplication by $m(x)$" operator. So $H_\hbar(m) = L^2(\mathbb{R}^n, m^2(x)dx)$. The Sobolev norm is

$$\|u\|_{H_\hbar(m)} = \|u\|_{L^2(\mathbb{R}^n_x, m^2(x)dx)}.$$

Example. On the other hand, suppose $m = m(\xi)$ depends only on ξ and $m \in S(m)$. Then we have $\widehat{m}^W u = \mathcal{F}_\hbar^{-1}[m(\xi)\mathcal{F}_\hbar u(\xi)]$. So

$$\widehat{m}^W u \in L^2(\mathbb{R}^n) \Longleftrightarrow m(\xi)\mathcal{F}_\hbar u \in L^2(\mathbb{R}^n).$$

Moreover, the Sobolev norm is given by

$$\|u\|^2_{H_\hbar(m)} = (2\pi\hbar)^{-n}\|\mathcal{F}_\hbar u\|^2_{L^2(\mathbb{R}^n_\xi, m^2(\xi)d\xi)}.$$

Example. In particular if $m(x, \xi) = \langle \xi \rangle^s$. Then

$$H^s_\hbar := H_\hbar(\langle \xi \rangle^s) = \left\{ u \in \mathscr{S}' \mid \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\mathcal{F}_\hbar u(\xi)|^2 d\xi < +\infty \right\}$$

and the Sobolev norm is explicitly given by

$$\|u\|^2_s = \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\mathcal{F}_\hbar u(\xi)|^2 d\xi.$$

Note that in the case $s = k$ is an nonnegative integer, H^k is the usual Sobolev space that we are familiar with:

$$H^k = \{u \in \mathscr{S}' \mid \|u\|^2_k := \sum_{|\alpha| \leq k} \|D^\alpha u\|^2_{L^2} < +\infty\}. \tag{2}$$

2. Semiclassical PsDO acting on the generalized Sobolev spaces

¶ Semiclassical PsDO acting on the generalized Sobolev spaces.

Just as in the previous example, Δ can be defined on $H^2(\mathbb{R}^n)$, we can prove

Proposition 2.1. *Suppose $m \geq 1$. For any $a \in S(m)$, there exists $h_0 > 0$ such that for any $\hbar \in (0, \hbar_0)$, the map $\widehat{a}^W : \mathscr{S} \to \mathscr{S}$ can be extended to a bounded linear operator $\widehat{a}^W : H_\hbar(m) \to L^2(\mathbb{R}^n)$.*

Proof. The proof is almost the same as above: We take $g = m$. by definition,

$$u \in H_\hbar(m) \Longleftrightarrow \widehat{m}^W u \in L^2(\mathbb{R}^n).$$

As before, $\widehat{a}^W \circ (\widehat{m}^W)^{-1}$ is semiclassical pseudodifferential operator with symbol in $S(1)$ and thus is a bounded linear operator on $L^2(\mathbb{R}^n)$. It follows

$$\|\widehat{a}^W u\|_{L^2} = \|\widehat{a}^W \circ (\widehat{m}^W)^{-1} \circ \widehat{m}^W u\|_{L_2} \leq C\|u\|_{H_\hbar(m)}.$$

□

As a consequence,

Corollary 2.2. *Assume $m \geq 1$, $a \in S(m)$ is real-valued. If $a+i$ is elliptic in $S(m)$, then $\widehat{a}^W : H_\hbar(m) \subset L^2 \to L^2$ is self-adjoint.*

Proof. Since a is real-valued, $\widehat{a}^W$ is symmetric. By ellipticity of $a+i$, the operator $\widehat{a}^W \pm i : H_\hbar(m) \to L^2$ has an inverse (which is a bounded linear operator on $L^2(\mathbb{R}^n)$) and thus is bijective. The conclusion follows. □

Another very important consequence is

Corollary 2.3. *Suppose $m \leq 1$ and suppose $a \in S(m)$ is elliptic. Then*

$$\widehat{a}^W : L^2(\mathbb{R}^n) \to H_\hbar(1/m),$$

and there exists $b \in S(1/m)$ so that $\widehat{b}^W : H_\hbar(1/m) \to L^2(\mathbb{R}^n)$ is the inverse of $\widehat{a}^W$.

Proof. The conclusion $\text{Image}(\widehat{a}^W) \subset H_\hbar(1/m)$ follows from the fact that $\widehat{(1/m)}^W \circ \widehat{a}^W$ has bounded symbol and thus is a bounded linear operator on L^2, so that

$$u \in L^2(\mathbb{R}^n) \Longrightarrow \widehat{(1/m)}^W \circ \widehat{a}^W u \in L^2(\mathbb{R}^n) \Longrightarrow \widehat{a}^W u \in H_\hbar(1/m).$$

By Lecture 14, $\widehat{a}^W$ admits a left inverse and a right inverse, namely, there exists $b, c \in S(1/m)$ so that at least on $\mathscr{S}$, we have

$$\widehat{a}^W \circ \widehat{b}^W = \text{Id}, \quad \widehat{c}^W \circ \widehat{a}^W = \text{Id}.$$

Since each operator maps $\mathscr{S}$ to $\mathscr{S}$, we get $\widehat{b}^W = \widehat{c}^W$ on $\mathscr{S}$. Since $m \leq 1$, we have $1/m \geq 1$. So both $\widehat{b}^W$ and $\widehat{c}^W$ can be extended to continuous linear operators from $H_\hbar(1/m)$ to $L^2(\mathbb{R}^n)$. Since $\mathscr{S}$ is dense in $H_\hbar(1/m)$, we conclude $\widehat{b}^W = \widehat{c}^W$. So $\widehat{b}^W$ is the inverse of $\widehat{a}^W$. This also implies $\widehat{a}$ is bijective onto $H_\hbar(1/m)$. □

¶ The generalized Sobolev spaces $H_\hbar(m)$ for any m (with $m \in S(m)$).

For an order function m with $m \in S(m)$, we define

Definition 2.4. The *generalized Sobolev space* associated with m is

$$H_\hbar(m) := \{u \in \mathscr{S}' \mid \widehat{m}^W u \in L^2(\mathbb{R}^n)\} \tag{3}$$

with the Sobolev norm

$$\|u\|_{H_\hbar(m)} = \|\widehat{m}^W u\|_{L^2(\mathbb{R}^n)}. \tag{4}$$

Note that this coincides with our earlier definition. Also note that for any m and any $a \in S(m)$, we have $\widehat{a}^W : \mathscr{S}' \to \mathscr{S}'$. So all the following expressions make sense as tempered distributions.

By definition, if $m \geq 1$, one may think of $H_\hbar(m)$ as a function space whose elements have more regularity than those in $H_\hbar(1) = L^2$, while if $m \leq 1$, one may think of $H_\hbar(m)$ as a "function space" whose elements have less regularity than those in $H_\hbar(1) = L^2$. We have just seeing how an operator of the form $\widehat{a}^W$ with $a \in S(m)$ will increase or decrease the regularity according to whether $m \geq 1$ or $m \leq 1$. It turns our that this is true for any two order functions:

Proposition 2.5. *Suppose m and m' are order functions on $\mathbb{R}^{2n}$. For any $a \in S(m)$, we have*

$$\widehat{a}^W \in \mathcal{L}(H_\hbar(m'), H_\hbar(m'/m)),$$

with the operator norm bound uniform in $\hbar$.

Proof. Since m' is elliptic in $S(m')$, we can find $b \in S(1/m')$ such that

$$\widehat{b}^W \circ \widehat{m'}^W = \mathrm{Id} \quad \text{on} \quad \mathscr{S}.$$

Since $(m'/m) \star a \star b \in S(1)$, we conclude that for any $u \in \mathscr{S}$,

$$\|\widehat{a}^W u\|_{H_\hbar(m'/m)} = \|\widehat{m'/m}^W \circ \widehat{a}^W \circ \widehat{b}^W \circ \widehat{m'}^W u\|_{L^2} \leq C\|u\|_{H_\hbar(m')},$$

where the constant $C = \|\widehat{m'/m}^W \circ \widehat{a}^W \circ \widehat{b}^W\|_{\mathcal{L}(L^2)}$ is uniform in $\hbar$. Since $\mathscr{S}$ is dense in each $H_\hbar(m)$ (prove this!), $\widehat{a}^W$ extends to a bounded linear operator from $H_\hbar(m'/m)$ to $H_\hbar(m'$ with operator norm bounded by the same constant C. □

As a consequence, we see

Corollary 2.6. *If $a \in S(m)$ is elliptic, then there exists $b \in S(1/m)$ such that for any m',*

$$\widehat{b}^W = (\widehat{a}^W)^{-1} \in \mathcal{L}(H_\hbar(m'/m), H_\hbar(m')).$$

Proof. We have see that there exist $b, c \in S(1/m)$ such that

$$\widehat{b}^W \circ \widehat{a}^W = \mathrm{Id} = \widehat{a}^W \circ \widehat{c}^W \quad \text{on} \quad \mathscr{S}.$$

It follows $\widehat{b}^W = \widehat{c}^W$ on $\mathscr{S}$. But $\widehat{b}^W, \widehat{c}^W \in \mathcal{L}(H_\hbar(m'/m), H_\hbar(m'))$, so they must coincide on $\mathcal{L}(H_\hbar(m'/m)$ since $\mathscr{S}$ is dense. □

With the "well-defined inverse" on suitable space, many earlier computations extends to all m. For example, Corollary 1.3 now holds for any m, m'. Also we can say

$$H_\hbar(m) = (\widehat{m}^W)^{-1} L^2(\mathbb{R}^n),$$

generalizing the similar formula at the top of page 2 for the ordinary Sobolev space $H^2(\mathbb{R}^n)$.

We list several other results, and leave the proofs as an happy exercise:

- The L^2 dual of $H_\hbar(m)$ is $H_\hbar(1/m)$.
- $\mathscr{S} = \cap_m H_\hbar(m)$ and $\mathscr{S}' = \cup_m H_\hbar(m)$.

¶ Compactness of $\widehat{a}^W$ for $a \in S(m)$.

We have the following result which generalize our earlier results for $L^2(\mathbb{R}^n)$:

Proposition 2.7. *Suppose m and m' are order functions on $\mathbb{R}^{2n}$. If*

$$\lim_{z\to\infty} m(z) = 0,$$

then for any $a \in S(m)$, the operator

$$\widehat{a}^W : H_\hbar(m') \to H_\hbar(m')$$

is compact.

Proof. The condition implies $m'/m \geq Cm'$ and thus

$$\widehat{a}^W : H_\hbar(m') \to H_\hbar(m'/m) \subset H_\hbar(m').$$

The conclusion is equivalent to the fact that the map

$$\widehat{m'}^W \circ a^W \circ (\widehat{m'}^W)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is compact, which is true because $m' \star a \star b \in S(m)$. □

Finally we remark that there exists Sobolev space version of Beals's theorem and Sharp Garding inequality, for "classical symbols", a subset of $S(\langle\xi\rangle^k)$ that consists of symbols $a = a(x,\xi)$ in $S(\langle\xi\rangle^k)$ so that for all multi-indices α and β, there exists a constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} \langle\xi\rangle^{m-|\beta|}. \tag{5}$$

Such symbols has the nice property that they are invariant under coordinate change and thus can be defined on manifolds.