LECTURE 17: EGOROV'S THEOREM

1. The propagator

¶ The propagator.

Let $m \geq 1$ be an order function, and let $q \in S(m)$ be a real-valued symbol function. Assume q is "almost elliptic" in the sense that there exists constants $C, c > 0$ such that

$$C + q \geq cm.$$

Since q is real-valued, we have

$$|q+i| = \sqrt{q^2+1} \geq \frac{1}{2}(q+1) > \frac{q+C}{2+2C} \geq \frac{c}{2+2C}m.$$

In other words, $q+i$ is elliptic in $S(m)$. According to Corollary 2.2 in Lecture 16, for $\hbar$ small enough, the operator

$$Q = \widehat{q}^W : H_\hbar(m) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is a densely-defined self-adjoint operator on $L^2(\mathbb{R}^n)$. So by the Stone's theorem that we cited in Lecture 8,

$$U(t) = e^{-itQ/\hbar} \tag{1}$$

is a strongly continuous one-parameter family of unitary operators on $L^2(\mathbb{R}^n)$ solving the equation

$$\begin{cases} \hbar D_t U(t) + QU(t) = 0, & t \in \mathbb{R} \\ U(0) = I. \end{cases} \tag{2}$$

The family $e^{-itQ/\hbar}$ is called the *propagators* generated by Q.

Note that $q + C$ is a positive-valued elliptic symbol in $S(m)$. It follows that for any $u \in S(m)$,

$$\|(Q+C)e^{-itQ/\hbar}u\|_{L^2} = \|e^{-itQ/\hbar}(Q+C)u\|_{L^2} = \|(Q+C)u\|_{L^2}.$$

As a consequence, $e^{-itQ/\hbar}$ is a bounded linear operator on $H_\hbar(m)$. [We will NOT say that $\|e^{-itP/\hbar}\|_{\mathcal{L}(H_\hbar(m))}$ equals one, since usually we will use another equivalent norm on $H_\hbar(m)$.]

We remark that in general the propagator $e^{-itP/\hbar}$ is NOT a semiclassical pseudodifferential operator, because it violates Beals's theorem (Check this!). It is a Fourier integral operator that we will study in more detail later.

¶ The "propagator" generated by t-dependent symbols.

For later purpose we extend the propagator defined above to t-dependent symbols. Now let $\{q_t\}_{t\in\mathbb{R}} \subset S(1)$ be a family of real-valued symbols that depend smoothly on t. Denote

$$Q(t) = \widehat{q_t}^W.$$

Let the operator $F(t)$ be the solution (which exists at least locally) to the system

$$\begin{cases} \hbar D_t F(t) + Q(t)F(t) = 0, & t \in \mathbb{R} \\ F(0) = I. \end{cases} \tag{3}$$

We extend the properties of the propagator $e^{-itQ/\hbar}$ to this time-dependent setting:

Theorem 1.1. *The solution $F(t)$ to the system (3) exists for all $t \in \mathbb{R}$ and is a unitary operator on $L^2(\mathbb{R}^n)$. Moreover, for any order function m, $F(t)$ is a bounded linear map from $H_\hbar(m)$ to $H_\hbar(m)$ for any t.*

Proof. Since $q_t \in S(1)$, the operator $Q(t)$ is bounded on L^2 (with uniform operator norm bound on closed intervals since $q(t)$ depends smoothly in t) and thus the solution (in the Banach space of bounded linear operators) exists for all t.

To prove $F(t)$ is unitary, we need to find out $F(t)^*$. Since q_t is real-valued, $Q(t)$ is formally self-adjoint, i.e. $Q(t)^* = Q(t)$. Take the adjoint of the system (3), we get

$$\begin{cases} \hbar D_t F(t)^* - F(t)^*Q(t) = 0, & t \in \mathbb{R} \\ F(0)^* = I. \end{cases}$$

It follows

$$\begin{aligned} \hbar D_t[F(t)^*F(t)] &= [\hbar D_t F(t)^*]F(t) + F(t)^*[\hbar D_t F(t)] \\ &= F(t)^*Q(t)F(t) - F(t)^*Q(t)F(t) \\ &= 0. \end{aligned}$$

So $F(t)^*F(t) = I$ since $F(0)^*F(0) = I$. On the other hand,

$$\begin{aligned} \hbar D_t[F(t)F(t)^* - I] &= -Q(t)F(t)F(t)^* + F(t)F(t)^*Q(t) \\ &= [F(t)F(t)^* - I, Q(t)]. \end{aligned}$$

This is a homogeneous ODE on $F(t)F(t)^* - I$, with initial condition

$$F(0)F(0)^* - I = 0.$$

It follows $F(t)F(t)^* = I$. So $F(t)$ is unitary on L^2.

For the second half of the theorem, without loss of generality we assume $m \in S(m)$. Since m is elliptic in $S(m)$, for $\hbar$ small enough the operator $(\widehat{m}^W)^{-1}$ is a pseudo-differential operator with symbol in $S(\frac{1}{m})$. Consider the operator

$$F_m(t) = \widehat{m}^W F(t)(\widehat{m}^W)^{-1}.$$

Then to prove the $\mathcal{L}(H_\hbar(m))$ boundedness of $F(t)$ is equivalent to prove the the $\mathcal{L}(L^2)$ boundedness of $F_m(t)$. Then $F_m(t)$ satisfies the equation

$$\hbar D_t F_m(t) = -\widehat{m}^W Q(t)F(t)(\widehat{m}^W)^{-1} = -Q_m(t)F_m(t)$$

for $Q_m(t) = \widehat{m}^W Q(t)(\widehat{m}^W)^{-1}$, with initial condition

$$F_m(0) = I.$$

Since $q_t \in S(1)$, the Weyl symbol of $Q_m(t) = \widehat{m}^W Q(t)(\widehat{m}^W)^{-1}$ is also in $S(1)$. So $Q_m(t)$ is a bounded linear operator on L^2. Hence by Gronwall's inequality,

$$\|F(t)\|_{\mathcal{L}(H_\hbar(m))} = \|F_m(t)\|_{\mathcal{L}(L^2)} \le e^{\int_0^t \|Q_m(s)\|_{\mathcal{L}(L^2)} ds} < +\infty.$$

□

Remark. Similar results hold for $q_t \in S(m)$ under a uniform ellipticity assumption. More precisely, suppose $m \ge 1$, suppose

$$q_t + C \ge m/C$$

for some constant $C > 0$. Moreover, assume the symbolic estimate holds uniformly for t-derivatives:

$$|\partial_t^k \partial_{x,\xi}^\alpha q_t| \le C_{k,\alpha} m.$$

Then the system (3) has a local solution which is invertible and maps $H_\hbar(m^k)$ into $H_\hbar(m^k)$ for any integer k. For details, c.f. Zworski, §10.1.3.

2. Egorov's theorem

¶ Hamiltonian flow.

Recall from Lecture 2 that associated to any smooth function $q(x,\xi)$ defined on $\mathbb{R}^{2n}$ one has a Hamiltonian flow $\{\rho_t\}_{t\in\mathbb{R}}$, that is, a family of diffeomorphisms

$$\rho_t = e^{t\Xi_q} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

that sends a point $z_0 = (x_0, \xi_0)$ to the point $z_t = \rho_t(z_0) = \gamma_{z_0}(t)$, where $\gamma = \gamma_{z_0}(t)$ is the unique integral curve of Ξ_q starting at z_0:

$$\gamma(0) = z_0, \quad \dot\gamma(t) = \Xi_q(\gamma(t)).$$

Here, Ξ_q is the Hamiltonian vector field generated by q,

$$\Xi_q = \sum_k \left(\frac{\partial q}{\partial \xi_k}\frac{\partial}{\partial x_k} - \frac{\partial q}{\partial x_k}\frac{\partial}{\partial \xi_k} \right).$$

As we have seen in Lecture 2, for any smooth symbol $a \in C^\infty(\mathbb{R}^{2n})$, if we denote $b_t = \rho_t^* a$, then

$$\dot b_t = \{q, b_t\}.$$

¶ A weak Egorov theorem.

Now let $a \in S(m)$ be a symbol, and let $b_t = \rho_t^* a$ be the "classical flow-out" of the symbol $a(x, \xi)$ along the flow ρ_t. A natural question is:

$$\boxed{\text{Question:}}\ \text{What is the Weyl quantization of } b_t?$$

Without any control on the flow the function $b_t = \rho_t^* a$ could be very bad. For simplicity we assume ρ is identity outside a compact set, this amounts to require that q is compactly supported (From the assumption on ρ we see that q is a constant outside a compact set. Since subtracting a constant from a Hamiltonian function will not change the flow, we may assume that constant is zero.) and in particular $q \in \mathscr{S} \subset S(1)$. The assumption also implies $b_t \in S(m)$ since $b_t = a$ outside a compact set. Now we may state the Egorov theorem that we mentioned in Lecture 1 in more precise form:

Theorem 2.1 (Weak Egorov's[1] theorem)**.** *Let $Q = \widehat{q}^W$. Under the previous assumptions, namely q is compactly supported and $a \in S(m)$, we have*

$$e^{itQ/\hbar}\widehat{a}^W e^{-itQ/\hbar} = \widehat{b_t}^W + O(\hbar^2),$$

where the estimate is uniform for $0 \le t \le T$, where T is any fixed time.

Remark. We need to explain our notation. For a $\hbar$-dependent bounded linear operator $A_\hbar \in \mathcal{L}(H_1, H_2)$, when we write $A_\hbar = O(\hbar)$, we mean $\|A_\hbar\|_{\mathcal{L}(H_1,H_2)} = O(\hbar)$. For this theorem, we can take $H_1 = H_\hbar(m)$ and $H_2 = L^2$.

Proof. Since $e^{itQ/\hbar}$ is bounded on each $H_\hbar(m)$, it is enough to prove

$$a^W - e^{-itQ/\hbar}\widehat{b_t}^W e^{itQ/\hbar} = O(\hbar^2).$$

Since $a = b_0$, we only need to prove

$$\frac{d}{dt}\left(e^{-itQ/\hbar}\widehat{b_t}^W e^{itQ/\hbar}\right) = O(\hbar^2),$$

i.e.

$$e^{-itQ/\hbar}\left(\widehat{\dot{b}_t}^W - \frac{i}{\hbar}\left[Q, \widehat{b_t}^W\right]\right)e^{itQ/\hbar} = O(\hbar^2).$$

Conjugating by the bounded operator $e^{itQ/\hbar}$ again, it is enough to prove

$$\widehat{\left(\dot{b}_t - \frac{i}{\hbar}(q \star b_t - b_t \star q)\right)}^W = O(\hbar^2),$$

[1] This is proved by Y.V. Egorov (1939-2008), a Russian-Soviet mathematician who specializes in differential equations. Another well-known Egorov theorem (in real analysis) was proven by D.F. Egorov (1869-1931), a Russian and Soviet mathematician known for significant contributions to the areas of differential geometry and mathematical analysis.

Since $\dot{b}_t = \{q, b_t\}$ and since (in Lecture 9-10)

$$q \star b_t - b_t \star q = \frac{\hbar}{i}\{q, b_t\} + O(\hbar^3),$$

the conclusion follows. □

The weak version of Egorov's theorem claims that when conjugated by the propagator, a semiclassical pseudodifferential operator $\widehat{a}^W$ gets converted to an operator which is semiclassically close to the semiclassical pseudodifferential operator whose symbol is the flow-out of a. So roughly speaking, "conjugation by the propagator" is the quantum analogue of "flow out by the classical flow". But we also want to know: what is the operator $e^{itQ/\hbar}\widehat{a}^W e^{-itQ/\hbar}$ itself? Is it still a semiclassical pseudodifferential operator. The answer is yes. [NOTE: as we have mentioned, the propagator $e^{itQ/\hbar}$ is NOT a semiclassical pseudodifferential operator.]

¶ The Time-dependent flow.

We will prove the theorem in more general setting, namely, we will consider the flow generated by a family of symbols q_t instead of generated by a single symbol q. We can repeat the construction of the Hamiltonian flow above: starting with a smooth family of symbols q_t, which, as explained above, will be assumed to be compactly supported in a fixed compact set. Consider the associated time-dependent Hamiltonian vector fields

$$\Xi_{q_t} = \sum_k \left(\frac{\partial q_t}{\partial \xi_k}\frac{\partial}{\partial x_k} - \frac{\partial q_t}{\partial x_k}\frac{\partial}{\partial \xi_k} \right).$$

Under the "compactly-supported" assumption, for any $z_0 = (x_0, \xi_0)$, Ξ_{q_t} admits a unique integral curve $\gamma = \gamma_{z_0}(t)$ starting at z_0,

$$\gamma(0) = z_0, \quad \dot{\gamma}(t) = \Xi_{q_t}(\gamma(t))$$

This again gives us a family of diffeomorphisms (which need not satisfy the group law as in the previous setting)

$$\rho_t = e^{t\Xi_H} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

that sends a point z_0 to the point $z_t = \rho_t(z_0) := \gamma_{z_0}(t)$. Again for any smooth symbol $a \in C^\infty(\mathbb{R}^{2n})$, if we denote $a_t = \rho_t^* a$, then

$$\dot{a}_t = \{q_t, a_t\}.$$

¶ Egorov's theorem.

Under the assumption that $\{q_t\}$ vanish outside a fixed compact set, we have $q(t) \in S(1)$ and thus we can apply Theorem 1.1 to $Q(t) = \widehat{q}_t^W$ to conclude that the operator equation

$$\begin{cases} \hbar D_t F(t) + Q(t)F(t) = 0, \quad 0 \le t \le T \\ F(0) = I. \end{cases}$$

has a unique solution F which is unitary on L^2 and is a bounded linear map from $H_\hbar(m)$ to $H_\hbar(m)$. In particular, by taking adjoint one gets

$$\hbar D_t[F(t)^{-1}] - F(t)^{-1}Q(t) = 0.$$

Now we can state the main theorem which tells us that the time-evolution of a semiclassical pseudodifferential operator $\widehat{a}^W$ is again a semiclassical pseudodifferential operator whose symbol, to the leading order, is the time-evolution of a:

Theorem 2.2 (Egorov's theorem)**.** *Suppose q_t ($t \in [0,T]$) is a smooth family of functions supported in a fixed compact set in $\mathbb{R}^{2n}$. Let m be an order function and suppose $a \in S(m)$. Let F be as described above. Then for $0 \le t \le T$, we have*

$$F(t)^{-1}\widehat{a}^W F(t) = \widehat{b_t}^W \tag{4}$$

for some symbol $b_t \in S(m)$ of the form $b_t = \rho_t^* a + O(\hbar)$.

Idea of the proof: We want to show that

$$B(t) := F(t)^{-1}\widehat{a}^W F(t)$$

is a semiclassical pseudodifferential operator. Of course we need to apply Beals's theorem. But $B(t)$ itself is not easy to handle, since $F(t)$ is not a semiclassical pseudodifferential operator. The idea is to approximate $B(t)$ to arbitrary order $O(\hbar^k)$ by pseudodifferential operators.

- First extend the weak Egorov theorem to the time-dependent setting, so that $B(t)$ is $\hbar$-close to the pseudodifferential operator $B_0(t) = \widehat{\rho_t^* a}^W$.
- Iteratively find pseudodifferential operator $B_k(t)$ that is $\hbar^{k+1}$-close to $B(t)$.
- Apply Beals's theorem to both $B_k(t)$ and $B_k(t) - B(t)$.

Proof.

Step 1: Extend the weak Egorov theorem to the time-dependent setting.

Denote $A = \widehat{a}^W$, $B_0(t) = \widehat{\rho_t^* a}^W$ and define

$$B(t) = F(t)^{-1}AF(t).$$

We want to show $B(t) - B_0(t) = O(\hbar)$.

We notice that by definition,

$$B(t) - B_0(t) = F(t)^{-1}(A - F(t)B_0(t)F(t)^{-1})F(t).$$

So we first study $F(t)^{-1}B_0(t)F(t)$. Since $\partial_t \rho_t^* a = \{q_t, \rho_t^* a\}$, we have

$$\hbar D_t B_0(t) = \frac{\hbar}{i}\widehat{\partial_t \rho_t^* a}^W = \frac{\hbar}{i}\widehat{\{q_t, \rho_t^* a\}}^W = [Q(t), B_0(t)] + E_0(t), \tag{5}$$

where $E_0(t) = \widehat{e_{0,t}}^W$ for some $e_{0,t} = O(\hbar^2)$. $e_{0,t}$ is compactly supported since it can be expressed as sums of products of derivatives of q_t and $\rho_t^* a$. So $e_{0,t} \in \hbar^2 \mathscr{S}$, and

$E_0(t)$ is a bounded operator on L^2 with operator norm $O(\hbar^2)$. It follows

$$\begin{aligned}\hbar D_t[F(t)B_0(t)F(t)^{-1}] &= [\hbar D_t F(t)]B_0(t)F(t)^{-1} + F(t)[\hbar D_t B_0(t)]F(t)^{-1} + F(t)B_0(t)\hbar D_t[F(t)^{-1}] \\ &= -Q(t)F(t)B_0(t)F(t) + F(t)([Q(t), B_0(t)] \\ &\qquad + E_0(t))F(t)^{-1} + F(t)B_0(t)F(t)^{-1}Q(t) \\ &= F(t)^{-1}E_0(t)F(t)\end{aligned}$$

is bounded on L^2 with operator norm $O(\hbar^2)$. So

$$F(t)B_0(t)F(t)^{-1} = A + \frac{i}{\hbar}\int_0^t F(s)E_0(s)F(s)^{-1}ds = A + O(\hbar).$$

So we conclude $B(t) - B_0(t) = O(\hbar)$.

Step 2: Construct $\hbar$-PsDO $B_k(t)$ such that $B(t) - B_k(t) = O(\hbar^k)$.

We just constructed $B_0(t)$ satisfying

$$\begin{cases} \hbar D_t B_0(t) = [Q(t), B_0(t)] + E_0(t) \\ B_0(0) = A. \end{cases}$$

such that $B(t) - B_0(t) = O(\hbar)$. To improve the error, we will construct a sequence of semiclassical pseudodifferential operators $B_k(t)$ so that for each k,

$$\begin{cases} \hbar D_t B_k(t) = [Q(t), B_k(t)] + E_k(t) \\ B_k(0) = A. \end{cases} \tag{6}$$

with

$$E_k(t) = \widehat{e_{k,t}}^W$$

for some $e_{k,t} \in \hbar^{k+2}\mathscr{S}$. Once this is done, the same computation gives

$$\hbar D_t[F(t)B_k(t)F(t)^{-1}] = F(t)E_k(t)F(t)^{-1} \in O(\hbar^{k+2}).$$

Again after integrating in t we will get

$$B(t) = B_k(t) - \frac{i}{\hbar}F(t)^{-1}\left[\int_0^t F(s)E_k(s)F(s)^{-1}ds\right]F(t), \tag{7}$$

which implies

$$B(t) = B_k(t) + O(\hbar^{k+1}). \tag{8}$$

We solve (6) by induction. Assume we have finished k. Then we try to find B_{k+1} of the form $B_{k+1} = B_k - C_{k+1}$ which solves (6) for $k+1$, where $C_{k+1}(t) := \widehat{c_{k+1,t}}^W \in O(\hbar^{k+1})$. For this purpose we first find the equation that C_{k+1} should satisfy. It is not hard to write down it:

$$hD_t C_{k+1}(t) = [Q(t), C_{k+1}(t)] + E_k(t) + O(\hbar^{k+3}).$$

It is thus very clear that to the leading order, the symbol c_{k+1} of C_{k+1} should satisfy the equation

$$\hbar D_t c_{k+1} = \{q_t, c_{k+1}\} + e_k(t).$$

This is an inhomogeneous evolution equation whose solution is given by Duhamel's principle,

$$c_{k+1,t} = \frac{i}{\hbar}\rho_t^* \int_0^t (\rho_s^{-1})^* e_{k,s} ds.$$

So we take $c_{k+1,t}$ as above, and define $C_{k+1}(t) := \widehat{c_{k+1,t}}^W \in O(\hbar^{k+1})$.. It is not hard to compute

$$\begin{aligned}\hbar D_t C_{k+1}(t) &= \left(\frac{\hbar}{i}\partial_t\left(\frac{i}{\hbar}\rho_t^* \widehat{\int_0^t (\rho_s^{-1})^* e_{k,s} ds}\right)\right)^W \\ &= \left(\frac{\hbar}{i}\{q_t, \widehat{c_{k+1,t}}\} + e_{k,t}\right)^W \\ &= [Q(t), C_{k+1}(t)] + E_k(t) - E_{k+1}(t)\end{aligned}$$

where $E_{k+1}(t) = \widehat{e_{k+1,t}}^W$ for some $e_{k+1,t} \in \hbar^{k+3}\mathscr{S}$. Hence

$$B_{k+1}(t) := B_k(t) - C_{k+1}(t)$$

is what we want.

Step 3: Prove $B(t)$ itself is a $\hbar$-PsDO with $B(t) - B_0(t) = O(\hbar)$.

According to Beals's theorem, it is enough to show that for any linear functions $l_1, \cdots, l_M$,

$$\mathrm{ad}_{\widehat{l_1}^W} \cdots \mathrm{ad}_{\widehat{l_M}^W}(B(t) - B_0(t)) = O(\hbar^{M+1}).$$

We will prove that for any k

$$\mathrm{ad}_{\widehat{l_1}^W} \cdots \mathrm{ad}_{\widehat{l_k}^W}(B(t) - B_k(t)) = O(\hbar^{k+1}), \tag{9}$$

so that for any M,

$$\begin{aligned}&\mathrm{ad}_{\widehat{l_1}^W} \cdots \mathrm{ad}_{\widehat{l_M}^W}(B(t) - B_0(t)) \\ =&\mathrm{ad}_{\widehat{l_1}^W} \cdots \mathrm{ad}_{\widehat{l_M}^W}(B(t) - B_M(t)) + \mathrm{ad}_{\widehat{l_1}^W} \cdots \mathrm{ad}_{\widehat{l_M}^W}(B_M(t) - B_0(t)) \\ =&O(\hbar^{M+1}).\end{aligned}$$

To prove (9), we notice that for any linear function l, the equation (7) implies

$$\mathrm{ad}_{\widehat{l}^W}(B(t) - B_k(t)) = -\frac{i}{\hbar}\int_0^t \mathrm{ad}_{\widehat{l}^W}\left[F(t)^{-1}F(s)E_k(s)F(s)^{-1}F(t)\right] ds.$$

Since $E_k(t) = \widehat{e_{k,t}}^W$ for $e_{k,t} \in \hbar^{k+2}\mathscr{S}$, we have

$$\frac{i}{\hbar}\mathrm{ad}_{\widehat{l}^W} E_k(s) = O(\hbar^{k+1}).$$

Similar expression holds for more $ad_{\widehat{l_j}^W}$'s. So it remains to prove

$$\mathrm{ad}_{\widehat{l_1}^W} \cdots \mathrm{ad}_{\widehat{l_k}^W} F(t) = O(1).$$

Recall

$$\mathrm{ad}_A(BC) = (\mathrm{ad}_A B)C + B(\mathrm{ad}_A C)$$

and

$$\mathrm{ad}_A B = -B(\mathrm{ad}_A B^{-1})B.$$

Applying $\mathrm{ad}_{\widehat{l^W}}$ to the equation

$$\hbar D_t F(t) + Q(t)F(t) = 0$$

we get

$$\hbar D_t(\mathrm{ad}_{\widehat{l^W}} F(t)) + (\mathrm{ad}_{\widehat{l^W}} Q(t))Q(t) + Q(t)\mathrm{ad}_{\widehat{l^W}} F(t) = 0$$

So

$$\begin{aligned}\hbar D_t(F(t)^{-1}\mathrm{ad}_{\widehat{l^W}} F(t)) &= F(t)^{-1}Q(t)\mathrm{ad}_{\widehat{l^W}} F(t) - F(t)^{-1}[(\mathrm{ad}_{\widehat{l^W}} Q(t))F(t) + Q(t)\mathrm{ad}_{\widehat{l^W}} F(t)] \\ &= -F(t)^{-1}[\mathrm{ad}_{\widehat{l^W}} Q(t)]F(t),\end{aligned}$$

which moves the operator $\mathrm{ad}_{\widehat{l^W}}$ from $F(t)$ to $Q(t)$! Since $Q(t)$ is a pseudodifferential operator, we have $\mathrm{ad}_{\widehat{l^W}} Q(t) = \hbar Q_1$ for $Q_1 = O(\hbar)$. Hence

$$\begin{aligned}\mathrm{ad}_{\widehat{l^W}} F(t) &= F(t)\frac{i}{\hbar}\int_0^t \hbar D_s(F(s)^{-1}\mathrm{ad}_{\widehat{l^W}} F(s))ds \\ &= i\int_0^t F(t)F(s)^{-1}Q_1(s)F(s)ds = O(1).\end{aligned}$$

Apply the method iteratively, we get the same estimate for more $\mathrm{ad}_{\widehat{l^W}}$'s. This completes the proof. □

Remark. For the proof we can see that the estimate can be done only for $t \in [0, T]$, where T is a fixed interval. For example, if instead fixing T we take $t = \hbar^{-1}$, then we will get extra $\hbar^{-1}$ factor in the estimates and the proof will fail. It turns out that Egorov theorem can be extend to longer time, namely,

$$t \in (0, -C\log\hbar).$$

The time $-C\log\hbar$ is called *Ehrenfest time*. For details about the Egorov theorem up to Ehrenfest time, c.f. Zworski, §11.4.