LECTURE 18: SEMI-CLASSICAL PsDOs WITH CLASSICAL SYMBOLS

1. Classical symbols

¶ Change of coordinates in $T^*\mathbb{R}^n$.

We are aiming at extending the definition of semiclassical pseudodifferential operators acting on $L^2(\mathbb{R}^n)$ to operators acting on functions on manifolds. For this purpose we need to study how a semiclassical pseudodifferential operator will change under a coordinate change.

Let's start by studying how coordinates will change in phase space. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, which represents a coordinate change

$$x \rightsquigarrow y = f(x)$$

in configuration space. Then the induced coordinate change in phase space will have the form

$$(x, \xi) \rightsquigarrow (y = f(x), \eta = \eta(x, \xi)).$$

To figure out the new cotangent variables η, let's recall the definition of cotangent variable ξ: Given any 1-form α at a point x, the coordinate ξ of α is determined by the equation

$$\alpha = \xi_1 dx_1 + \cdots + \xi_n dx_n.$$

Similarly in the new coordinate system (y, η), one must have

$$\alpha = \eta_1 dy_1 + \cdots + \eta_n dy_n.$$

It follows that

$$\eta^T dy = \xi^T dx = \xi^T (\frac{\partial x}{\partial y}) dy.$$

So the new cotangent coordinates η is given by

$$\eta = ([\frac{\partial y}{\partial x}]^{-1})^T \xi.$$

¶ Change of symbols under coordinate change in $T^*\mathbb{R}^n$.

Now we study the following problem: if we change the coordinate system in the configuration space (and thus change the coordinate system in the phase space as described above), how will the symbol change?

For this purpose, let's suppose $a = a(x, \xi)$. Consider the coordinate change

$$(x, \xi) \rightsquigarrow \left(y(x), ([\frac{\partial y}{\partial x}]^{-1})^T \xi \right)$$

Let $\tilde{a}$ be the function $\tilde{a}(y,\eta) = a(x,\xi)$. Suppose $\tilde{a} \in S(m)$. Then

$$\partial_{x_j} a(x,\xi) = \left(\frac{\partial y_k}{\partial x_j}\right) \partial_{y_k}\tilde{a}(y,\eta) + \frac{\partial}{\partial x_j}\left[([\frac{\partial y}{\partial x}]^{-1})^T \xi\right]_k \partial_{\eta_k}\tilde{a}(y,\eta).$$

So in general $a(x,\xi)$ will not be a symbol in $S(m)$, even if we pose a boundedness assumption on the Jacobian matrix $\left(\frac{\partial y_k}{\partial x_j}\right)$ of the coordinate change in configuration space (which is not a very restrictive assumption since we will mainly work on compact manifolds). The bad term is the ξ appeared in the expression. Even if we are working on compact manifold, the cotangent variable ξ is still unbounded, so that $\partial_x a$ is no longer bounded by m. (On the other hand, it is easy to see that the ξ-derivatives of a behaves as nice as η-derivatives of $\tilde{a}$ if we assume the boundedness of the Jacobian. So the ξ-term is essentially the only bad term.)

¶ Classical symbols.

To solve this problem, we will have to restrict ourselves to order functions of special form. In view of the computation above, we are naturally lead to study those symbols which will lost one ξ-order after taking each ξ-derivative.

Definition 1.1. A *classical symbol* of order $m \in \mathbb{R}$ is a function $a \in S(\langle\xi\rangle^m)$ so that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|} \tag{1}$$

for all multi-indices α and β. As usual, the symbol a is allowed to be $\hbar$-dependent, in which case we require the constant $C_{\alpha,\beta}$ to be uniform for $\hbar \in (0,\hbar_0)$.

We will denote this class S^m, and denote

$$\Psi^m = \{\widehat{a}^W \mid a \in S^m\}.$$

We will also denote

$$S^{-\infty} = \bigcap_{m\in\mathbb{Z}} S^m, \qquad S^{\infty} = \bigcup_{m\in\mathbb{Z}} S^m.$$

and

$$\Psi^{-\infty} = \bigcap_{m\in\mathbb{Z}} \Psi^m, \qquad \Psi^{\infty} = \bigcup_{m\in\mathbb{Z}} \Psi^m.$$

For later purpose, we list several simple properties of classical symbols whose proofs are quite obvious:

Proposition 1.2. *We have*

1. *If $a \in S^m$, then $\partial_x^\alpha \partial_\xi^\beta a \in S^{m-|\beta|}$.*
2. *If $a \in S^{m_1}, b \in S^{m_2}$, then $ab \in S^{m_1+m_2}$ and $\{a,b\} \in S^{m_1+m_2-1}$.*
3. *$\mathscr{S} \subset S^{-\infty}$.*

¶ Invariance of classical symbols.

Now we prove that the classical symbols is invariant under coordinate change (under the boundedness assumption on the derivatives of the Jacobian):

Theorem 1.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with*

$$|\partial^\alpha f| \le C_\alpha, |\partial^\alpha f^{-1}| \le C_\alpha \tag{2}$$

for all multi-indices α. Then for any classical symbol $a \in S^m$, its "pull back"

$$\widetilde{f}^* a(x,\xi) := a\left(f^{-1}(x), \left(\frac{\partial f}{\partial x}\bigg|_{f^{-1}(x)}\right)^T \xi\right)$$

is also a classical symbol in S^m.

Proof. If we write $c(x,\xi) = a(f(x),\xi)$ then $\widetilde{f}^* a = c\left(x, (\partial f_x)^T \xi\right)$, and thus the function $\partial_x^\alpha(\widetilde{f}^* a)$ has the form

$$\sum_{|\gamma|+|\sigma|\le|\alpha|, |\sigma|=|\rho|} C_{\gamma\sigma\rho}(\partial_x^\gamma \partial_\xi^\sigma c)\xi^\rho$$

It follows

$$\partial_x^\alpha \partial_\xi^\beta \left(\widetilde{f}^* a\right) = \sum_{|\gamma|+|\sigma|\le|\alpha|, |\sigma|=|\rho|, |\kappa|=|\lambda|, |\nu|=\beta, \nu\ge\kappa, \rho\ge\lambda} C_{\gamma\sigma\rho\nu\kappa\lambda}(\partial_x^\gamma \partial_\xi^{\sigma+\nu-\kappa} c)\xi^{\rho-\lambda}.$$

So we conclude

$$\left|\partial_x^\alpha \partial_\xi^\beta \left(\widetilde{f}^* a\right)\right| \le \sum C\langle\xi\rangle^{m-|\sigma|-|\nu|+|\kappa|}\langle\xi\rangle^{|\rho|-|\lambda|} \le C\langle\xi\rangle^{m-|\beta|}.$$

□

Remark. In applications the diffeomorphism f can be taken to be a orientation-preserving diffeomorphism from a bounded region to another bounded region, which could be extended to a diffeomorphism on $\mathbb{R}^n$ that is the identity map outside a compact set, so that the condition (2) hold automatically.

¶ Polyhomogeneous symbols S^m_{phg}.

In literature there is another widely used symbol class, the class of *polyhomogeneous symbols*, which is by definition the space of symbols $a \in S^m$ of the form

$$a \sim \sum_{k=-\infty}^{m} a_k,$$

where a_k is a symbol which is a degree k homogeneous function with respect to ξ, namely,

$$a_k(x, \lambda\xi) = \lambda^k a_k(x,\xi).$$

One can prove that this class is also invariant under coordinate change, and establish symbolic calculus for this class as before. Note that pseudodifferential operators with polyhomogeneous symbols already contains all differential operators.

2. Semiclassical Pseudodifferential operators with classical symbols

¶ Symbolic calculus for classical symbols.

Note that $S^m \subset S(\langle\xi\rangle^m)$, so the formula we have proven for $S(\langle\xi\rangle^m)$ can be applied to classical symbols. Moreover, due to the the improvement under differentiation in ξ, there are new features for the symbolic calculus for symbols in S^m.

Theorem 2.1. *Suppose $a \in S^{m_1}$ and $b \in S^{m_2}$, then $a \star b \in S^{m_1+m_2}$. Moreover,*

$$a\star b-\sum_{k=0}^{N}\frac{1}{k!}\frac{(i\hbar)^k}{2^k}(D_\xi\cdot D_y-D_x\cdot D_\eta)^k\left[a(x,\xi)b(y,\eta)\right]\Big|_{y=x,\eta=\xi}\in \hbar^{N+1}S^{m_1+m_2-N-1}. \tag{3}$$

Proof. We have already known $a \star b \in S(\langle\xi\rangle^{m_1+m_2})$ and

$$\begin{aligned}a\star b(x,\xi) &= e^{\frac{i\hbar}{2}(D_\xi\cdot D_y-D_x\cdot D_\eta)}(a(x,\xi)b(y,\eta))\Big|_{y=x,\eta=\xi}\\ &\sim \sum_{k=0}^{\infty}\frac{1}{k!}\frac{(i\hbar)^k}{2^k}\left(D_\xi\cdot D_y-D_x\cdot D_\eta\right)^k\left[a(x,\xi)b(y,\eta)\right]\Big|_{y=x,\eta=\xi}.\end{aligned}$$

To estimate the remainder, we need to use the remainder formula for Taylor's expansion, namely the left hand side of (3) equals

$$R_N=C_N\int_0^1(1-t)^N e^{\frac{i\hbar}{2}t(D_\xi\cdot D_y-D_x\cdot D_\eta)}\hbar^{N+1}\left(D_\xi\cdot D_y-D_x\cdot D_\eta\right)^{N+1}\left[a(x,\xi)b(y,\eta)\right]\Big|_{y=x,\eta=\xi}dt.$$

Since $(D_\xi\cdot D_y-D_x\cdot D_\eta)^{N+1}$ encounters exactly $N+1$ derivatives of ξ and η, we have

$$c_k:=\frac{1}{k!}\frac{(i\hbar)^k}{2^k}(D_\xi\cdot D_y-D_x\cdot D_\eta)^k\left[a(x,\xi)b(y,\eta)\right]\Big|_{y=x,\eta=\xi}\in\hbar^k S^{m_1+m_2-k}$$

and

$$\left(D_\xi\cdot D_y-D_x\cdot D_\eta\right)^{N+1}\left[a(x,\xi)b(y,\eta)\right]\in\sum_{k=0}^{N+1}S(\langle\xi\rangle^{m_1-k}\langle\eta\rangle^{m_2-(N+1-k)})$$

On the other hand, we have seen in Lecture 10 (Theorem 3.2) that the operator $e^{\frac{i\hbar}{2}t(D_\xi\cdot D_y-D_x\cdot D_\eta)}$ preserve such spaces. It follows

$$a\star b-\sum_{k=0}^{N}\frac{1}{k!}\frac{(i\hbar)^k}{2^k}(D_\xi\cdot D_y-D_x\cdot D_\eta)^k\left[a(x,\xi)b(y,\eta)\right]\Big|_{y=x,\eta=\xi}\in\hbar^{N+1}S(\langle\xi\rangle^{m_1+m_2-N-1}).$$

Now for any α,β, we have

$$\partial_x^\alpha\partial_\xi^\beta(a\star b)=\sum_{k=0}^{|\beta|-1}\partial_x^\alpha\partial_\xi^\beta c_k+\partial_x^\alpha\partial_\xi^\beta R_{|\beta|-1}\in\sum_{k=0}^{|\beta|-1}\hbar^k S^{m_1+m_2-k-|\beta|}+\hbar^{|\beta|}S(\langle\xi\rangle^{m_1+m_2-|\beta|})$$

which implies
$$|\partial_x^\alpha \partial_\xi^\beta (a \star b)| \le C_{\alpha,\beta} \langle \xi \rangle^{m_1+m_2-|\beta|}.$$
where $C_{\alpha,\beta}$ is uniform for $\hbar \in (0, \hbar_0)$. So $a \star b \in S^{m_1+m_2}$.

The proof of (3) is similar: To prove $|\partial_x^\alpha \partial_\xi^\beta R_N| \le C_{\alpha,\beta}\langle \xi \rangle^{m_1+m_2-|\beta|}$, we split R_N into two parts, with first $|\beta|-1$ terms handled term by term in $S^{m_1+m_2-N-1-k-|\beta|}$, and the remaining terms handled together as an element in $S(\langle \xi \rangle^{m_1+m_2-N-1-|\beta|})$. $\square$

Remark. Similar property holds for other t-quantizations.

As a consequence, we have

Corollary 2.2. *If $A \in \Psi^{m_1}, B \in \Psi^{m_2}$, then*

1. $AB \in \Psi^{m_1+m_2}$.
2. $[A, B] \in \hbar \Psi^{m_1+m_2-1}$.

¶ Mapping properties for operators in Ψ^m.

For operators $A \in \Psi^m$, it is natural to consider the action of A on Sobolev spaces $H_\hbar(m)$ with $m = \langle \xi \rangle^s$. We denote
$$H_\hbar^s := H_\hbar(\langle \xi \rangle^s) = \{u \in \mathscr{S}' \mid \langle P \rangle^s \in L^2\}.$$
Since $S^m \subset S(\langle \xi \rangle^m)$, we immediately get the following version of Calderon-Vaillancourt theorem for classical symbols:

Proposition 2.3. *Suppose $a \in S^m$. Then $\widehat{a}^W \in \mathcal{L}(H_\hbar^s, H_\hbar^{s-m})$.*

Note that if $A \in \Psi^m$, then
$$\mathrm{ad}_{\mathrm{Q_j}} A = -\hbar \widehat{D_{\xi_j} a}^W \quad \text{and} \quad \mathrm{ad}_{\mathrm{P_j}} A = -\widehat{D_{x_j} a}^W.$$
So for any $i_1, \cdots, i_M, j_1, \cdots, j_N \in \{1, \cdots, n\}$, one has
$$\mathrm{ad}_{Q_{i_1}} \cdots \mathrm{ad}_{Q_{i_M}} \mathrm{ad}_{P_{j_1}} \cdots \mathrm{ad}_{P_{i_N}} A = (-1)^{M+N} \hbar^M \partial_x^\alpha \partial_\xi^\beta a \in \hbar^M \Psi^{m-M}$$
and thus
$$\|\mathrm{ad}_{Q_{i_1}} \cdots \mathrm{ad}_{Q_{i_M}} \mathrm{ad}_{P_{j_1}} \cdots \mathrm{ad}_{P_{i_N}} A\|_{\mathcal{L}(H_\hbar^{s+m}, H_\hbar^{s+M})} = O(\hbar^M).$$
Conversely we also has the following Beals's characterization for operators in Ψ^m:

Theorem 2.4 (Beals theorem for Ψ^m)**.** *A continuous linear operator $A : \mathscr{S} \to \mathscr{S}'$ is in class Ψ^m if and only if there exists $s \in \mathbb{R}$ and for any $i_1, \cdots, i_M, j_1, \cdots, j_N \in \{1, \cdots, n\}$, one has*
$$\|\mathrm{ad}_{Q_{i_1}} \cdots \mathrm{ad}_{Q_{i_M}} \mathrm{ad}_{P_{j_1}} \cdots \mathrm{ad}_{P_{i_N}} A\|_{\mathcal{L}(H_\hbar^{s+m}, H_\hbar^{s+M})} = O(\hbar^M).$$

¶ Sharp Gårding inequality for Ψ^m.

Suppose $a \in S^m$ then $\widehat{a}^W$ maps L^2 to $H_\hbar^{-m}$. And map H^m to L^2

We also have the following sharp Gårding inequality for classical symbols:

Theorem 2.5 (Sharp Gårding inequality for classical symbols)**.** *Suppose $a \in S^m$ is a real-valued symbol such that $a \geq 0$. Then there exists $C \geq 0$ such that for any $u \in \mathscr{S}$,*

$$\langle \widehat{a}^W u, u\rangle \geq -C\hbar \|u\|^2_{H_\hbar^{(m-1)/2}}$$

Proof. We first observe that it is enough to prove the theorem for $m = 0$. Suppose this is done, then for any other $m \in \mathbb{R}$ and $a \in S^m$, we let

$$b = \langle \xi\rangle^{-m/2} \star a \star \langle \xi\rangle^{-m/2}.$$

Then by Theorem 2.1, $b \in S^0$, and thus

$$\langle \widehat{a}^W u, u\rangle = \langle \widehat{b}^W \circ \widehat{\langle\xi\rangle^{m/2}}^W u, \widehat{\langle\xi\rangle^{m/2}}^W u\rangle \geq -C\hbar \| \widehat{\langle\xi\rangle^{m/2}}^W u\|_{H_\hbar^{-1/2}} = -C\hbar\|u\|^2_{H_\hbar^{m-1/2}}.$$

So in what follows we assume $m = 0$, and prove

$$\langle \widehat{a}^W u, u\rangle \geq -C\hbar\|u\|^2_{H_\hbar^{-1/2}} = \frac{-C\hbar}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} \langle\xi\rangle^{-1} |\mathcal{F}_\hbar u(\xi)|^2 d\xi.$$

Note that in Lecture 15 we already proved (for nonnegative symbol $a \in S^0 \subset S(1)$) $\langle \widehat{a}^W u, u\rangle \geq -C\hbar\|u\|^2_{L^2}$. To improve the estimate from L^2-norm to $H^{-1/2}$-norm, we will use dyadic decomposition in frequency space to decompose a and thus $\widehat{a}^W$, and estimate term by term:

Lemma 2.6 (Dyadic partition of unity)**.** *There exists $\psi_0 \in C_0^\infty(\mathbb{R}^n)$ and $\psi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, both with values in $[0, 1]$, such that*

$$1 = \psi_0(\xi) + \sum_{j=0}^{\infty} \psi(2^{-j}\xi).$$

Proof. Exercise. □

In what follows we fix such an dyadic POU and denote $\psi_j(\xi) = \psi(2^{-j}\xi)$. Then

$$a(x, \xi) = \sum_{j=0}^{\infty} \psi_j(\xi)a(x, \xi) = \sum_{j=0}^{\infty} a_j(x, 2^{-j}\xi),$$

where $a_0(x, \xi) = \psi_0(\xi)a(x, \xi)$ and $a_j(x, \xi) = \psi(\xi)a(x, 2^j\xi)$ for $j \geq 1$. One has

$$|\partial_x^\alpha \partial_\xi^\beta a_j| \leq C_{\alpha\beta} 2^{j|\beta|} \max_{\xi\in\text{supp}(\psi)} \langle 2^j\xi\rangle^{-|\beta|} \leq \widetilde{C}_{\alpha\beta}.$$

i.e. $a_j \in S(1)$ with $S(1)$-seminorm bounds uniform in j. It follows that

$$\langle \widehat{a_j}^W u, u\rangle \geq -C\hbar\|u\|^2_{L^2(\mathbb{R}^n)},$$

where C is independent of j.

Now we use the decomposition

$$\widehat{a}^W = \sum_j \widehat{a}_j^{\,W}|_{\hbar=2^{-j}\hbar}.$$

We choose $\widetilde{\psi}_0 \in C_0(\mathbb{R}^n)$ and $\widetilde{\psi} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, both with values in $[0,1]$, such that

$$\widetilde{\psi} = 1 \text{ on } \mathrm{supp}(\psi), \quad \widetilde{\psi}_0 = 1 \text{ on } \mathrm{supp}(\psi_0),$$

and denote $\widetilde{\psi}_j(\xi) = \widetilde{\psi}(2^{-j}\xi)$ as before. Then $\widetilde{\psi} = 1$ on $\mathrm{supp}(a_j)$ and

$$\widehat{\widetilde{\psi}_j}^W \widehat{a}_j^{\,W}|_{\hbar=2^{-j}\hbar}\widehat{\widetilde{\psi}}^W = \left(\widehat{\widetilde{\psi}}^W \widehat{a}_j^{\,W}\widehat{\widetilde{\psi}}^W\right)\Big|_{\hbar=2^{-j}\hbar} = \widehat{a}_j^{\,W}|_{\hbar=2^{-j}\hbar} + O(\hbar_j^\infty).$$

It follows that for any $u \in \mathscr{S}$,

$$\begin{aligned}\langle \widehat{a}^W u, u\rangle &= \sum_j \langle \widehat{a}_j^{\,W}|_{\hbar=2^{-j}\hbar} u, u\rangle \\ &\ge \sum_j \left(\langle \widehat{a}_j^{\,W}|_{\hbar=2^{-j}\hbar}\widehat{\widetilde{\psi}}^W u, \widehat{\widetilde{\psi}}^W u\rangle - O(\hbar^N 2^{-jN})\|u\|_{H_\hbar^{-1/2}}\right) \\ &\ge -C\hbar \sum_j 2^{-j}\|\widehat{\widetilde{\psi}}^W u\|_{L^2}^2 - O(\hbar^N)\|u\|_{H_\hbar^{-1/2}},\end{aligned}$$

Now we use the following fact: there exists $C>0$ such that for any $u \in \mathscr{S}$,

$$\sum_{j=0}^{\infty} 2^{-j}\|\widehat{\widetilde{\psi}_j}^W u\|_{L^2}^2 \le C\|u\|_{H_\hbar^{-1/2}}^2. \tag{4}$$

Proof: Recall from the definition (Lecture 16) that

$$\|u\|_{H_\hbar^{-1/2}}^2 = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} \langle \xi\rangle^{-1}|\mathcal{F}_\hbar u(\xi)|^2 d\xi.$$

Since

$$\sum_j 2^{-j}\|\widehat{\widetilde{\psi}_j}^W u\|_{L^2}^2 = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\sum_j 2^{-j}|\widetilde{\psi}_j|^2|\mathcal{F}_\hbar u|^2 d\xi,$$

Since

$$2^{-j} \le C\langle \xi\rangle^{-1}, \quad \xi \in \mathrm{supp}(\widetilde{\psi}_j)$$

and since for any ξ, there are at most finitely many different j's so that $\widetilde{\psi}_j(\xi) \neq 0$ and there is at least one j with $\widetilde{\psi}_j(\xi) = 0$, we conclude

$$\sum_j 2^{-j}|\widetilde{\psi}_j(\xi)| \le C\langle \xi\rangle^{-1}, \quad \xi \in \mathrm{supp}(\widetilde{\psi}_j)$$

It follows

$$\langle \widehat{a}^W u, u\rangle \ge -C\hbar\|u\|_{H_\hbar^{-1/2}}^2$$

□

Remark.

- Sharp Garding inequality also holds for other t-quantizations, in which case one only need to replace $\langle \widehat{a}^W u, u\rangle$ by $\mathrm{Re}\langle \widehat{a}^t u, u\rangle$.
- One also has the stronger *Fefferman-Phong inequality*:
$$\langle \widehat{a}^W u, u\rangle \geq -C\hbar^2 \|u\|^2_{H_\hbar^{(m-2)/2}}.$$