LECTURE 19: DIFFERENTIAL OPERATORS ON MANIFOLDS

1. Differential operators on manifolds

We are aiming at extending the definition of semiclassical pseudodifferential operators from $\mathbb{R}^n$ to manifolds. Let's start by the simplest class of pseudodifferential operators: the differential operators. For simplicity we consider $\hbar = 1$ only. One can easily extend to the semiclassical setting.

¶ Differential operators under coordinate change.

Let's assume U, V are open sets in $\mathbb{R}^n$ and let

$$f : U \subset \mathbb{R}^n_x \to V \subset \mathbb{R}^n_y$$

be a diffeomorphism. We can easily "transplant" a differential operator defined for x-functions to a differential operator defined for y-functions via f: If

$$P = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha \tag{1}$$

is a differential operator acting on $C^\infty(\mathbb{R}^n_x)$, then when restricted to U, P is also a differential operator $P|_U$ acting on $C^\infty(U)$, and $P|_U$ induces a differential operator $\widetilde{P}$ acting on $C^\infty(V)$ as follows: for any $u \in C^\infty(V)$, we just define

$$\widetilde{P}u := (f^{-1})^* P|_U f^* u.$$

Let's calculate $\widetilde{P}$ in coordinates: for any $u = u(y) \in C^\infty(V)$ we have

$$(f^*u)(x) = u(f(x))$$

and thus

$$[P|_U f^* u](x) = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha [u(f(x))].$$

Since

$$\partial_{x^i}[u(f(x))] = \frac{\partial y^j}{\partial x^i} (\partial_{y^j} u)(f(x)),$$

by induction it is easy to get

$$\partial_x^\alpha [u(f(x))] = \left[\left(\frac{\partial y}{\partial x} \right)^T \partial_y \right]^\alpha u(f(x)) + \text{l.o.t.},$$

where l.o.t. denotes terms that encounter less ∂_y-derivatives on u. It follows

$$\widetilde{P}u(y) = \sum_{|\alpha| = m} a_\alpha(f^{-1}(y)) \left[\left(\frac{\partial y}{\partial x} \right)^T \partial_y \right]^\alpha u(y) + \text{l.o.t.} \tag{2}$$

¶ Gluing differential operators on manifolds.

Now suppose M is a smooth manifold, $\{(\varphi_\alpha, U_\alpha, V_\alpha, x^1_\alpha, \cdots, x^n_\alpha)\}$ is a coordinate chart. For simplicity we assume M is compact, and the coordinate chart is finite. Recall that a function u defined on M is *smooth* if for any α,

$$u \circ \varphi_\alpha^{-1}$$

is smooth. This can be expressed in another way: if we have smooth functions $u_\alpha \in C^\infty(U_\alpha)$, (or equivalently, smooth functions $u_\alpha \circ \varphi_\alpha^{-1} \in C^\infty(V_\alpha)$, and if

$$u_\alpha \circ \varphi_\alpha^{-1} = (u_\beta \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\alpha^{-1}) \qquad \text{on} \quad \varphi_\alpha(U_\alpha \cap U_\beta),$$

then we can glue all these u_αs defined on U_αs, (or equivalently, glue all these $u_\alpha \circ \varphi_\alpha^{-1}$ defined on V_αs,) to *one* smooth function u defined on M: we just let

$$u(x) := u_\alpha(x)$$

for $x \in U_\alpha$. The above condition tells us that $u_\alpha = u_\beta$ on $U_\alpha \cap U_\beta$.

Now suppose $P_\alpha : C^\infty(V_\alpha) \to C^\infty(V_\alpha)$ be differential operators defined on V_αs,

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \subset V_\alpha \to \varphi_\beta(U_\alpha \cap U_\beta) \subset V_\beta$$

be the coordinate transition diffeomorphism. Assume that

$$(\varphi_{\alpha\beta}^{-1})^* P_\alpha|_{\varphi_\alpha(U_\alpha \cap U_\beta)} \varphi_{\alpha\beta}^* = P_\beta|_{\varphi_\beta(U_\alpha \cap U_\beta)} \quad \text{on} \quad C^\infty(\varphi_\beta(U_\alpha \cap U_\beta)).$$

Then we can "glue" P_α's via $\varphi_{\alpha\beta}$'s to get a differential operator on M: for any $u \in C^\infty(M)$ and $x \in U_\alpha \subset M$, we just let

$$Pu(x) := \varphi_\alpha^* P_\alpha((\varphi_\alpha^{-1})^* u)(x).$$

We check this P is well-defined: if $x \in U_\alpha \cap U_\beta$, then

$$\begin{aligned}
\varphi_\beta^* P_\beta((\varphi_\beta^{-1})^* u)(x) &= \varphi_\beta^* P_\beta|_{\varphi_\beta(U_\alpha \cap U_\beta)}((\varphi_\beta^{-1})^* u)(x) \\
&= \varphi_\beta^* \left[(\varphi_{\alpha\beta}^{-1})^* P_\alpha|_{\varphi_\alpha(U_\alpha \cap U_\beta)} \varphi_{\alpha\beta}^*\right] ((\varphi_\beta^{-1})^* u)(x) \\
&= \varphi_\beta^* \left[(\varphi_\beta^*)^{-1} \varphi_\alpha^* P_\alpha|_{\varphi_\alpha(U_\alpha \cap U_\beta)} (\varphi_\alpha^{-1})^* \varphi_\beta^*\right] ((\varphi_\beta^{-1})^* u)(x) \\
&= \varphi_\alpha^* P_\alpha|_{\varphi_\alpha(U_\alpha \cap U_\beta)} (\varphi_\alpha^{-1})^* u(x).
\end{aligned}$$

In the above constructions, the most important property we used to glue local functions or local differential operators to global ones is the locality of functions or differential operators themselves: in the case of differential operators, it is crucial that we can restrict a differential operator P on an open subset U to a differential operator P_1 on its open subset U_1; moreover, this restriction is "universal" in the sense that if P_2 is the restriction of P_1 onto an open subset U_2 of U_1, then P_2 is also the restriction of P onto the open subset U_2 of U.

¶ Differential operators on manifolds: an abstract definition.

Here is another way to express the locality of differential operators: for a differential operator P, the values of the function Pu on an open set U depends only on the values of u on U. Equivalently,

Definition 1.1. We say a linear operator $P : C^\infty(M) \to C^\infty(M)$ is *local* if for any $u \in C^\infty(M)$,

$$\operatorname{supp}(Pu) \subset \operatorname{supp}(u). \tag{3}$$

It is this "locality" that allows us to "glue" differential operators defined on local charts to a differential operator on the whole manifold: By definition it is easy to see that if P is a local operator on M, and if $U \subset M$ is an open subset, then the "restriction operation"

$$P|_U u := (Pu)|_U$$

defines a "restricted operator" $P|_U : C^\infty(U) \to C^\infty(U)$. Moreover, such restricted operators satisfies the property that for any open sets $U_1 \subset U$,

$$(P|_U)|_{U_1} = P|_{U_1}.$$

Now we can give an abstract definition of a differential operator on a smooth manifold:

Definition 1.2. Let M be a smooth manifold. A *differential operator* on M of order at most m is a local linear operator $P : C^\infty(M) \to C^\infty(M)$ such that when restricted to each coordinate chart $\{\varphi_\alpha, U_\alpha, V_\alpha, x^1, \cdots, x^n\}$, the operator

$$P_\alpha := (\varphi_\alpha^{-1})^* \circ P|_{U_\alpha} \circ \varphi_\alpha^*$$

is a differential operator on V_α of order at most m (namely, is of the form (1)).

Example. Any smooth vector field V on M is a differential operator of order 1. Conversely one can prove (exercise): any differential operator of order 1 on M has the form $V + m_f$, where V is a vector field, and f is a smooth function and m_f is the operator "multiplication by f" (which is a differential operator of order 0).

Example. In general, if V_i's are a finite collection of smooth vector fields, then

$$P = \sum_{0 \le k \le m} V_{j_1} \cdots V_{j_k}$$

is a differential operator on M of order at most m (where $k = 0$ represents a multiplication operator).

Conversely, at least for compact manifold, we can write any differential operator in this form. To see this, we just use a partition of unity subordinate to a coordinate covering, so that in each coordinate chart P has the form (1).

¶ Distributions and Sobolev spaces on manifolds.

In what follows we assume (M, g) is a compact Riemannian manifold, so that there is a well-defined Riemannian volume form using which we can define $L^2(M)$. (We can also develop the theory without a Riemannian metric, in which case we can use the space of *half densities*). As in the Euclidean case, one can define, for each non-negative integer k, the Sobolev space $H^k(M)$ by

(4)
$$H^k(M) = \{u \in L^2(M) \mid V_1 \cdots V_k u \in L^2(M) \text{ for all smooth vector fields } V_1, \cdots V_k\}.$$

Since M is compact, one can choose a family of vector fields $W_1, \cdots, W_N$ on M that span T_xM at each point x. The Sobolev norm on $H^k(M)$ is defined to be

$$\|u\|_{H^k(M)} = \left(\sum_{l=0}^{k} \sum_{1\leq\alpha_j\leq N} \|W_{\alpha_1} \cdots W_{\alpha_l} u\|^2_{L^2(M)}\right)^{1/2}.$$

while the semi-classical Sobolev norm on $H^k(M)$ is defined to be

$$\|u\|_{H^k_\hbar(M)} = \left(\sum_{l=0}^{k} \sum_{1\leq\alpha_j\leq N} \hbar^{2l} \|W_{\alpha_1} \cdots W_{\alpha_l} u\|^2_{L^2(M)}\right)^{1/2}.$$

To define Sobolev spaces $H^k(M)$ for negative k, one has to extend the conception of distributions to manifolds. Again the idea is to quite simple: we pull-back everything to Euclidian space via coordinate charts. Suppose $(\varphi_\alpha, U_\alpha, V_\alpha)$ is a coordinate chart. Then given any $u : C^\infty(M) \to \mathbb{C}$, we want to "transplant" u to be a linear functional $\widetilde{u}$ on $\mathscr{S}(\mathbb{R}^n)$ via the chart map, so that we say u is a distribution if the induced linear map $\widetilde{u}$ is an element in $\mathscr{S}'$:

Definition 1.3. Let M be a smooth compact manifold. We say a linear map $u : C^\infty(M) \to \mathbb{C}$ is a *distribution* on M if for every coordinate chart $(\varphi_\alpha, U_\alpha, V_\alpha)$ and every $\chi \in C_0^\infty(V_\alpha)$, the mapping defined for $\varphi \in \mathscr{S}(\mathbb{R}^n)$ by

$$\varphi \mapsto u(\gamma^*(\chi\varphi)) \tag{5}$$

belongs to $\mathscr{S}'(\mathbb{R}^n)$. The space of distributions on M is denoted by $\mathscr{D}'(M)$.

Remark. In the case of noncompact manifolds, one can also define the space of distributions in a similar way. A more rigorous way: first define a topology on $C_0^\infty(M)$, then realize $\mathscr{D}'(M)$ as the dual space of $C_0^\infty(M)$. Here is how we define such a topology on $C_0^\infty(M)$: first we can always write $M = \cup_n \text{int}(K_n)$, where each K_n is compact and $K_n \subset \text{int}(K_{n+1})$ for any n. Since each K_n is compact, it is contained in finitely many coordinate charts. Using coordinate charts we can define a locally convex topology[1] on $C_0^\infty(\text{int}(K_n))$ via local semi-norms (c.f. Lecture 4). Now we get a sequence of locally convex topological spaces $C_0^\infty(\text{int}(K_n))$, so that

[1] A topological vector space is called *locally convex* if the origin has a neighborhood basis consisting of convex sets.

each $C_0^\infty(\text{int}(K_n))$ is a topological subspace of $C_0^\infty(\text{int}(K_{n+1}))$. Finally we define a topology on $C_0^\infty(M) = \cup_n C_0^\infty(\text{int}(K_n))$ to be the finest locally convex topology so that each inclusion $\iota_n : C_0^\infty(\text{int}(K_n)) \hookrightarrow C_0^\infty(M)$ is continuous. Such a topology is known as the *strict inductive limit topology*, which turns $C_0^\infty(M)$ into an *LF space*. For details of this construction, c.f. Reed-Simon Vol 1, Section 5.4.

By locality, if P is a differential operator, then P maps $C_0^\infty(M)$ to $C_0^\infty(M)$. So by duality, P maps $\mathscr{D}'(M)$ to $\mathscr{D}'(M)$. Now one can define

$$H^{-k}(M) = \text{span}\{u \in \mathscr{D}'(M) \mid u = V_1 \cdots V_l f \text{ for } f \in L^2(M), 0 \le l \le k\} \tag{6}$$

with Sobolev norm

$$\|u\|_{H^{-k}(M)} = \inf\{\sum_\alpha \|f_\alpha\|_{L^2(M)} \mid u = \sum_\alpha W_{\alpha_1} \cdots W_{\alpha_l} f_\alpha\}.$$

or semiclassical Sobolev norm

$$\|u\|_{H_\hbar^{-k}(M)} = \inf\{\sum_\alpha \|f_\alpha\|_{L^2(M)} \mid u = \sum_\alpha \hbar^l W_{\alpha_1} \cdots W_{\alpha_l} f_\alpha\}.$$

Obviously if P is a differential operator of order m, then P maps $H^k(M)$ to $H^{k-m}(M)$.

2. Symbolic calculus of differential operators

¶ Differential operators on manifolds: principle symbols.

For differential operators on $\mathbb{R}^n$, say, the operator

$$P = \sum_{|\alpha| \le m} a_\alpha D^\alpha,$$

we can define its full Kohn-Nirenberg symbol to be

$$\sigma_{KN}(P)(x,\xi) := \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha,$$

which is, of course, a function on $T^*\mathbb{R}^n = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$. Similarly one can define the Weyl symbol of $P = \sum_{|\alpha|\le m} a_\alpha D^\alpha$, which is given by (c.f. Lecture 9)

$$\sigma_W(P)(x,\xi) = e^{\frac{i}{2}\partial_x\cdot\partial_\xi}\left(\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha\right) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha + \text{l.o.t.}, \tag{7}$$

where l.o.t. represents a polynomial in ξ whose degree is at most $m-1$. So although $\sigma_{KN}(P)(x,\xi) \neq \sigma_W(P)(x,\xi)$, they are both polynomials in ξ of degree m, and their leading terms are the same. (Of course the same conclusion holds for any t-quantization.)

It is natural to ask: can we define the Kohn-Nirenberg or Weyl symbol for differential operators on manifolds? Unfortunately the answer is no, because the full symbol, as a function on T^*M, is not well-defined. Let me remind you that the coordinate change on the cotangent space T^*M induced by a coordinate change $x \rightsquigarrow y = f(x)$ on the base manifold is given by

$$(x, \xi) \rightsquigarrow \left(y = f(x), \eta = ([\frac{\partial y}{\partial x}]^{-1})^T \xi \right).$$

So if a function σ is well-defined on T^*M, and it has the form $\sigma(x, \xi)$ in one chart (x, ξ), then it should has the form

$$\sigma \left(f^{-1}(y), (\frac{\partial y}{\partial x})^T \eta \right)$$

in the other chart (y, η). However, as we have seen, if $P = \sum a_\alpha(x) D_x^\alpha$ in one chart, then in the other chart, it has the form $\widetilde{P}$ given by (2). Because of the complicated nature of l.o.t. (which also contains derivatives of the coordinate change diffeomorphism), we have

$$\sigma_{KN}(P)(x, \xi) \neq \sigma_{KN}(\widetilde{P})(f^{-1}(y), (\frac{\partial y}{\partial x})^T \eta)$$

However, if one stare at the formula (2), one can easily see that the terms with $|\alpha| = m$ do satisfy the correct "change of variable" formula! Thus we define

Definition 2.1. The *principal symbol* of a differential operator P of order m on a smooth manifold M is defined to be the smooth function $\sigma_m(P) \in C^\infty(T^*M)$ so that on a coordinate chart $(\varphi, U, x^1, \cdots, x^n\}$, if P has the form $P = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha$, then

$$\sigma_m(P)(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha.$$

Remark. In view of (2), the principal symbol of a differential operator is a well-defined smooth function on T^*M. Moreover, we don't need to distinguish the principal symbol in different t-quantizations, since, as we just explained in (7), they are all the same at the "principal" level!

Example. Let V be a smooth vector field V, viewed as a differential operator of order 1. So if we write $V = \sum a_j D_j$ in a local chart, and recall $\xi = \sum \xi_j dx^j$, we get, then a local computation yields

$$\sigma_1(V)(x, \xi) = \sum a_j \xi_j = \sum a_j \xi(\partial_j) = \xi(\sum_j a_j \partial_j)$$

In other words, we get

$$\sigma_1(V)(x, \xi) = \xi(iV_x) = i\xi(V_x).$$

More generally, the principal symbol of $P = \sum V_{j_1} \cdots V_{j_k}$ on M is $\sum \sigma(V_{j_1}) \cdots \sigma(V_{j_k})$.

Example. The most important example is the Laplace-Beltrami operator Δ_g on a Riemannian manifold (M, g), which locally has the form

$$\Delta_g = -\frac{1}{\sqrt{|g|}} \sum \partial_i (g^{ij} \sqrt{|g|} \partial_j).$$

It is a second order differential operator on M with principle symbol

$$\sigma(\Delta_g)(x, \xi) = \sum g^{ij} \xi_i \xi_j = |\xi|_g^2.$$

¶ Symbolic calculus for differential operators.

Let's denote the set of differential operators on M of order no more than m by $\mathcal{D}^m(M)$. Then the principal symbol gives a map

$$\sigma_m : \mathcal{D}^m(M) \to C^\infty(T^*M).$$

Then by definition, we have

Proposition 2.2. *If $P \in \mathcal{D}^m(M)$ and $\sigma_m(P) = 0$, then $P \in \mathcal{D}^{m-1}(M)$.*

Since differential operators are special cases of pseudodifferential operators, according to the symbolic calculus for pseudodifferential operators, we have

Proposition 2.3. *If $P \in \mathcal{D}^{m_1}(M), Q \in \mathcal{D}^{m_2}(M)$, then*

(1) $P \circ Q \in \mathcal{D}^{m_1}(M)$ *and*
$$\sigma_{m_1+m_2}(P \circ Q) = \sigma_{m_1}(P) \sigma_{m_2}(Q).$$
(2) $[P, Q] \in \mathcal{D}^{m_1+m_2-1}(M)$ *and* [2]
$$\sigma_{m_1+m_2-1}([P, Q]) = \{\sigma_{m_1}(P), \sigma_{m_2}(Q)\}.$$

Remark. Although the full symbol of a differential operator is not well-defined on T^*M, there is a *sub-principal symbol* which is well-defined on T^*M. More precisely, suppose P is a differential operator on M which has the form $P = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ in local charts. As we have seen, the principal symbol $\sigma_m(P) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha$ is well defined on T^*M. The next term,

$$\sigma_{m-1}(P)(x, \xi) = \sum_{|\alpha|=m-1} a_\alpha(x) \xi^\alpha$$

is only locally defined and is not well defined on T^*M. However, one can check that the function (exercise)

$$\sigma_{\mathrm{sub}}(P)(x, \xi) := \sigma_{m-1}(P) + \frac{i}{2} \sum_j \frac{\partial^2}{\partial x^j \partial \xi^j} \sigma_m(x, \xi)$$

is a well-defined function on T^*M. It is called the sub-principal symbol of P.

[2]Here, $\{\cdot, \cdot\}$ is the Poisson bracket on T^*M which locally has the form $\{f, g\} = \sum (\partial_{\xi_j} f \partial_{x_j} g - \partial_{x_j} f \partial_{\xi_j} g)$. One can check that it is a well-defined function on T^*M.