LECTURE 20: SEMI-CLASSICAL PsDOs ON MANIFOLDS

1. Semiclassical PsDOs under coordinate change

¶ Pseudolocality.

As we have mentioned several times, we want to extend the theory of semiclassical pseudodifferential operators from $\mathbb{R}^n$ to smooth manifolds. So in some sense, we have to study how to "glue" local pseudodifferential operators to global ones. As we have seen last time, for differential operators it is easy to get global operators from local ones, because of the *locality* of differential operators. In general, pseudodifferential operators does not satisfy locality. For example, one can look at the Dirichlet-to-Neumann operator Λ in PSet 2, which is by definition the map that sends the Dirichlet boundary value of a Harmonic function on $\mathbb{R}^{n+1}_+$ to its Neumann boundary value. It is easy to find $f \in C^\infty(\mathbb{R}^n)$ with $f \equiv 0$ on an open subset U in $\mathbb{R}^n$, while $\Lambda(f) \neq 0$ on U.

Fortunately, although locality fails, we do have a weaker version of locality for pseudodifferential operators, namely, the *pseudolocality*. Roughly speaking, locality tells us that the value $Au(x)$ at a point x is determined by the values $u(y)$ for y near x, while pseudolocality tells us that the value $Au(x)$ is determined, *modulo* $O(\hbar^\infty)$ *(which is negligible in semiclassical analysis)*, by the values $u(y)$ for y near x. To see the pseudolocality of $A = \widehat{a}^W$, we just start with the definition of $\widehat{a}^W$, namely

$$\widehat{a}^W u(x) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi} a(\frac{x+y}{2},\xi)u(y)dy$$

and notice that away from the diagonal, namely in the region $|x-y| > C$ (which can be produced by multiplying u by a cut-off function that is supported away from the point x), one can produce as many $\hbar$'s as we want via integration by parts using

$$e^{\frac{i}{\hbar}(x-y)\cdot\xi} = \hbar\frac{(x-y)\cdot D_\xi}{|x-y|^2}e^{\frac{i}{\hbar}(x-y)\cdot\xi}.$$

Pseudolocality can also be explained as follows. Suppose $a \in S(m)$ and suppose $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^n)$ are two cut-off functions such that

$$\mathrm{supp}(\chi_1)\cap\mathrm{supp}(\chi_2) = \emptyset.$$

Then (c.f. PSet 3)

$$\chi_1\widehat{a}^W\chi_2 = O(\hbar^\infty).$$

So if we take χ_1 to be a cut-off function supported near x (which equals 1 in a small neighborhood of x) and take χ_2 to be a cut-off function supported away, then the effect of the value of u in the support of χ_2 is negligible in computing $\widehat{a}^W u(x)$.

¶ Semiclassical PsDOs under coordinate change.

So we still want to study the change of semiclassical pseudodifferential operator under coordinate change. Again we start with a diffeomorphism

$$f : U \subset \mathbb{R}^n_x \to V \subset \mathbb{R}^n_y$$

where we always assume

$$|\partial^\alpha f|, |\partial^\alpha f^{-1}| \le C_\alpha, \quad \forall \alpha,$$

and again we want to use f to "transplant", now a pseudodifferential operator P defined for x-variables to a pseudodifferential operator for y-variables. However, we can't define $\widetilde{P}$ to be the operator $(f^{-1})^* P|_U f^*$, since pseudodifferential operators are not local and thus $P|_U$ makes no sense. The solution to this issue is straightforward: instead of localize the operator, we "globalize" the functions. How do we "globalize" a locally defined function? Multiply it by a cut-off function! More precisely, suppose $\chi \in C_0^\infty(V)$, then we have maps

$$f^* M_\chi : \mathscr{S}(\mathbb{R}^n_y) \to C_0^\infty(U) \subset C_0^\infty(\mathbb{R}^n_x) \subset \mathscr{S}(\mathbb{R}^n_x)$$

and

$$M_\chi (f^{-1})^* : \mathscr{S}(\mathbb{R}^n_x) \subset C^\infty(\mathbb{R}^n_x) \to C_0^\infty(\mathbb{R}^n_y) \subset \mathscr{S}(\mathbb{R}^n_y).$$

which extends to maps[1]

$$f^* M_\chi : \mathscr{S}'(\mathbb{R}^n_y) \to \mathscr{S}'(\mathbb{R}^n_x)$$

and

$$M_\chi (f^{-1})^* : \mathscr{S}'(\mathbb{R}^n_x) \to \mathscr{S}'(\mathbb{R}^n_y).$$

So given any pseudodifferential operator P on $\mathbb{R}^n_x$, which, for simplicity, we assume its Kohn-Nirenberg symbol is a, namely $P = \widehat{a}^{KN}$ (for a in some symbol class), and given any cut-off function $\chi \in C_0^\infty(V)$, we can define $\widetilde{P}_\chi : \mathscr{S}(\mathbb{R}^n_y) \to \mathscr{S}(\mathbb{R}^n_y)$ (and $\widetilde{P}_\chi : \mathscr{S}'(\mathbb{R}^n_y) \to \mathscr{S}'(\mathbb{R}^n_y)$) by

$$\widetilde{P}_\chi = M_\chi (f^{-1})^* P f^* M_\chi. \tag{1}$$

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[1]Given a tempered distribution $u \in \mathscr{S}'(\mathbb{R}^n_y)$ and a compactly supported function $\chi = \chi(y)$, we can extend M_χ to $M_\chi : \mathscr{S}'(\mathbb{R}^n_y) \to \mathscr{S}'(\mathbb{R}^n_y)$ defined by

$$\langle M_\chi u, \varphi \rangle := \langle u, M_\chi \varphi \rangle, \qquad \forall \varphi \in \mathscr{S}(\mathbb{R}^n_y).$$

The definition of pull-back of distribution is a bit complicated: Given any diffeomorphism $f : U \to V$, one has the pull-back map on functions, $f^* : C_0^\infty(V) \to C_0^\infty(U)$. By duality, we get a linear map $f_* : \mathscr{D}(U) \to \mathscr{D}(V)$, called the *push-forward*, defined by

$$\langle f_* u, \varphi \rangle := \langle u, f^* \varphi \rangle.$$

Moreover, it can be shown that the restriction of f_* to $C_0^\infty(U)$ is a continuous linear map from $C_0^\infty(U)$ to $C_0^\infty(V)$. By duality again, we get a *pull-back* map, now defined on distributions, $f^* : \mathscr{D}'(V) \to \mathscr{D}'(U)$. [For pull-back by submersions, c.f. Theorem 6.1.2 in Hörmander, Vol 1.]

¶ Semiclassical PsDOs under coordinate change: Schwartz symbols.

We have to check that the operator $\widetilde{P}_\chi$ defined above is a pseudodifferential operator, and calculate its symbol:

Theorem 1.1. *Suppose $a \in \mathscr{S}$ and $P = \widehat{a}^W$. Then the operator $\widetilde{P}_\chi$ defined by (1) is a pseudodifferential operator whose Kohn-Nirenberg symbol $b \in \mathscr{S}$ and has the asymptotic expansion*

$$b(y,\eta) \sim a\left(f^{-1}(y), (\partial f)^T\eta\right)\chi(y)^2 + \sum_{j\geq 1}\hbar^j b_j(y,\eta), \tag{2}$$

for some $b_j \in \mathscr{S}$.

Proof. By definition we have

$$\begin{aligned}
\widetilde{P}_\chi u(y) &= \frac{\chi(y)}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(f^{-1}(y)-z)\cdot\xi} a\left(f^{-1}(y),\xi\right)\chi(f(z))u(f(z))dzd\xi \\
&= \frac{\chi(y)}{(2\pi\hbar)^n}\int_{U\times\mathbb{R}^n} e^{\frac{i}{\hbar}(f^{-1}(y)-z)\cdot\xi} a\left(f^{-1}(y),\xi\right)\chi(f(z))u(f(z))dzd\xi \\
&= \frac{\chi(y)}{(2\pi\hbar)^n}\int_{V\times\mathbb{R}^n} e^{\frac{i}{\hbar}(f^{-1}(y)-f^{-1}(w))\cdot\xi} a\left(f^{-1}(y),\xi\right)\chi(w)u(w)|\det\partial f^{-1}|dwd\xi \\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \chi(y)e^{\frac{i}{\hbar}(f^{-1}(y)-f^{-1}(w))\cdot\xi} a\left(f^{-1}(y),\xi\right)\chi(w)u(w)|\det\partial f^{-1}(w)|dwd\xi.
\end{aligned}$$

So $\widehat{P}_\chi$ is an operator whose Schwartz kernel $K_{\widetilde{P}_\chi}$ is compactly supported (since the support is contained in $\mathrm{supp}(\chi)\times\mathrm{supp}(\chi)$). It follows (c.f. the computation on page 7 of Lecture 14) that $\widehat{P}_\chi$ is the Weyl quantization of the symbol

$$\int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}w\cdot\xi}K_{\widetilde{P}_\chi}(x+\frac{w}{2}, x-\frac{w}{2})dw$$

which can be shown to be Schwartz. To calculate the Kohn-Nirenberg symbol $b(y,\eta)$ of $\widehat{P}_\chi$, we use the oscillatory test (PSet 2) to get

$$\begin{aligned}
b(y,\eta) &= e^{-\frac{i}{\hbar}y\cdot\eta}\widehat{P}_\chi(e^{\frac{i}{\hbar}y\cdot\eta}) \\
&= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}[(f^{-1}(y)-f^{-1}(w))\cdot\xi+(w-y)\cdot\eta]}a(f^{-1}(y),\xi)\chi(y)\chi(w)|\det\partial f^{-1}(w)|dwd\xi \\
&=: \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}\varphi_{y,\eta}}a_y(\omega,\xi)d\omega d\xi.
\end{aligned}$$

So according to the lemma of stationary phase (Lecture 5),

$$b(y,\eta) \sim \sum_{d\varphi_{y,\eta}(p)=0} e^{\frac{i}{\hbar}\varphi_{y,\eta}(p)}e^{\frac{i\pi}{4}\mathrm{sgn}(d^2\varphi_{y,\eta}(p))}\frac{a_y(p)}{|\det d^2\varphi_{y,\eta}(p)|^{1/2}}\sum_j \hbar^j L_j(a_y)(p),$$

where L_j is a differential operator in w,ξ of order $2j$ and $L_0 = 1$.

So we calculate the critical points of the phase function

$$\varphi_{y,\eta}(w,\xi) = (f^{-1}(y) - f^{-1}(w))\cdot\xi + (w-y)\cdot\eta,$$

which is given by

$$\partial_\xi \varphi_{y,\eta} = 0 \Longrightarrow y = w$$

and

$$\partial_w \varphi_{y,\eta} = 0 \Longrightarrow \eta = (\partial f^{-1})^T \xi.$$

In other words, the phase function $\varphi_{y,\eta}$ admits a unique critical point

$$w = y, \quad \xi = (\partial f)^T \eta.$$

We can also calculate the Hessian of the phase function, which, at the critical point, has the form

$$d^2\varphi_{y,\eta} = \begin{pmatrix} d_w^2\varphi_{y,\eta} & -(\partial f^{-1})^T \\ -(\partial f^{-1}) & 0 \end{pmatrix}$$

and thus

$$\operatorname{sgn}(d^2\varphi_{y,\eta}) = 0^2 \quad \text{and} \quad |\det d^2\varphi_{y,\eta}|^{1/2} = |\det \partial f^{-1}(y)|.$$

Thus we conclude

$$b(y,\eta) \sim a\left(f^{-1}(y), (\partial f)^T\eta\right)\chi(y)^2\left[1 + \sum_{j\geq 1}\hbar^j L_j(a_y)(y,(\partial f)^T\eta)\right].$$

It remains to prove

$$b - a\left(f^{-1}(y), (\partial f)^T\eta\right)\chi(y)^2 - \sum_{1\leq j\leq k-1}\hbar^j b_j \in \hbar^k\mathscr{S}.$$

This can be proved inductively by a similar argument to the function $(y,\eta)^\alpha\partial^\beta_{y,\eta}b$. We omit the details.

[Another way to see $\widehat{P}_\chi$ is a semiclassical pseudodifferential operator: Assuming V is star-shaped. Let B be the $n\times n$ matrix whose (k,l)-entry is the integral $\int_0^1 \frac{\partial f_k^{-1}}{\partial x^l}(w+t(y-w))dt$. Then we have the formula (c.f. Lemma 2.6 in Lecture 5)

$$f^{-1}(y) - f^{-1}(w) = B(y-w)$$

and as a result, we can rewrite $\widehat{P}_\chi$ as

$$\begin{aligned}\widetilde{P}_\chi u(y) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}}\chi(y)e^{\frac{i}{\hbar}(y-w)\cdot B^T\xi}a\left(f^{-1}(y),\xi\right)\chi(w)u(w)|\partial f^{-1}(w)|dwd\xi \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}}\chi(y)e^{\frac{i}{\hbar}(y-w)\cdot\xi}a\left(f^{-1}(y),(B^T)^{-1}\xi\right)\chi(w)u(w)|\partial f^{-1}(w)|(B^T)^{-1}dwd\xi\end{aligned}$$

from which we can calculate the symbol.] □

[2]Reason: easy to show that the matrix is congruent to $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

¶ Semiclassical PsDOs under coordinate change: invariant symbols.

The theorem we just proved also holds for invariant symbols:

Theorem 1.2. *Suppose $a \in S^m$. Then the operator $\widetilde{P}_\chi$ defined by (1) is a pseudo-differential operator whose symbol $b \in S^m$ has the asymptotic expansion*

$$b(y,\eta) \sim a\left(f^{-1}(y), (\partial f)^T\eta\right)\chi(y)^2 + \sum_{j\geq 1} \hbar^j b_j(y,\eta), \tag{3}$$

for some $b_j \in S^{m-j}$.

Proof. The proof is lengthy and we will omit it. We need to apply a more complicated version of lemma of stationary phase, e.g. Theorem 7.7.7 in *Hörmander, The Analysis of Partial Differential Operators Vol I*. See also Theorem 18.1.7 in *Hörmander, The Analysis of Partial Differential Operators Vol III*. In fact, according to Theorem 18.1.7 in the book, one has the following somewhat different expression of b:

$$b(y,\eta) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha a(f^{-1}(y), (\partial f)^T\eta)(\chi(y))^2(\hbar D_z)^\alpha(e^{\frac{i}{\hbar}\rho_x(z)\cdot\eta})|_{z=x=f^{-1}(y)},$$

where $\rho_x(z) = f(x) - f(z) - \partial f(z)(x-z)$. It is crucial to notice that $\rho_x(x) = 0$ and $\partial\rho_x(x) = 0$, so that to produce one η-factor (and one $\hbar^{-1}$-factor) from $(\hbar D_z)^\alpha(e^{\frac{i}{\hbar}\rho_x(z)\cdot\xi})|_{z=x=f^{-1}(y)}$ you need at least two derivatives. As a result, the right hand side *is* an asymptotic expansion, but the α-term is NOT in $\hbar^{|\alpha|}S^{m-|\alpha|}$, but in $\hbar^{[|\alpha|/2]}S^{m-[|\alpha|/2]}$ instead. □

Remark. Usually an asymptotic expansion in S^m has the form

$$a(x,\xi) \sim \sum_{j=0}^\infty \hbar^j a_j(x,\xi),$$

where $a \in S^m$, and $a_j \in S^{m-j}$. In this setting, negligible symbols are symbols in $\hbar^\infty S^{-\infty}$.

2. Semiclassical pseudodifferential operator on manifolds

¶ Semiclassical pseudodifferential operator on manifolds.

Recall that for each m, the symbol class S^m consisting of symbols satisfying

$$|\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|} \text{ for all } \alpha,\beta$$

is invariant under coordinate changes in phase space, namely

$$(x,\xi) \rightsquigarrow \widetilde{f}(x,\xi) = \left(y = f(x), \eta = (\partial f^{-1})^T\xi\right),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism that has bounded derivatives (of any order), e.g. f is the identity map outside a compact set.

Now let M be a compact smooth manifold. Then from any coordinate chart $(\varphi_\alpha, U_\alpha, V_\alpha)$ we get a natural diffeomorphism

$$\widetilde{\varphi_\alpha^{-1}} : V_\alpha \times \mathbb{R}^n \to T^*U_\alpha$$

We define $S^m(T^*M)$ be the set of smooth functions $a \in C^\infty(T^*M)$ so that for any coordinate chart $(\varphi_\alpha, U_\alpha, V_\alpha)$,

$$\widetilde{\varphi_\alpha^{-1}}^* (a|_{T^*U}) \in S^m(V_\alpha \times \mathbb{R}^n). \tag{4}$$

According to the invariance property of the class S^m (Theorem 1.3 in Lecture 18), the set $S^m(T^*M)$ is well-defined.

We also need to introduce the class of *smoothing operators*. One can prove (exercise) that for $a \in S^m$, the Schwartz kernel k_a of $\widehat{a}^W$ is smooth off diagonal, i.e. $k_a = k_a(x, y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\})$. In general, we say $\widehat{a}^W$ is a *smoothing operator* if it has a smooth kernel $k_a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and the kernel satisfies

$$|\partial_x^\alpha \partial_y^\beta k_a(x, y)| \le C_{\alpha\beta N} (\frac{\hbar}{\langle x - y\rangle})^N$$

for all α, β and N. One can show that any smoothing operator is the Weyl quantization of a negligible symbol $a \in \hbar^\infty S^{-\infty}$.

On compact manifolds this condition can be a bit simpler: An operator A on a smooth manifold M is a *smoothing operator* if it has a smooth Schwartz kernel with $|\partial_x^\alpha \partial_y^\beta k_a(x, y)| \le C_{\alpha\beta N} \hbar^N$. The class of smoothing operators is denoted by $\hbar^\infty \Psi^{-\infty}(M)$.

Now we define

Definition 2.1. A linear operator $A = A_\hbar : C^\infty(M) \to C^\infty(M)$ is called a *(semi-classical) pseudo-differential operator* of order m on M if it has the form

$$A = \sum_j M_{\chi_j} \varphi_j^* \widehat{a_j}^W (\varphi_j^{-1})^* M_{\chi_j} + \hbar^\infty \Psi^{-\infty}(M)$$

for a finite collection of charts $\{\varphi_j, U_j, V_j\}$ and cut-off functions $\chi_j \in C_c^\infty(U_j)$, where $a_j \in S^m$.

We will denote the class of semiclassical pseudodifferential operator as define above by $\Psi^m(M)$. The following equivalent characterization is left as an exercise:

Proposition 2.2. *A linear operator $A = A_\hbar : C^\infty(M) \to C^\infty(M)$ is a (semi-classical) pseudo-differential operator of order m on M if and only if*

(1) *for each coordinate patch $(\varphi_\alpha, U_\alpha, V_\alpha)$, there exists a cut-off function χ and a symbol $a_{\alpha,\chi} \in S^m$ such that*

$$(\varphi_\alpha)^* M_\chi A M_\chi (\varphi_\alpha^{-1})^* = \widehat{a_{\alpha,\chi}}^W$$

(2) *A is* pseudolocal, *i.e. for any $\chi_1, \chi_2 \in C^\infty(M)$ with* $\mathrm{supp}\chi_1 \cap \mathrm{supp}\chi_2 = \emptyset$,

$$\chi_1 A \chi_2 \in \hbar^\infty \Psi^{-\infty}(M). \tag{5}$$

¶Mapping properties.

Now suppose M is a compact Riemannian manifold (or consider the half density bundle). Like the Euclidean case, for any $s \in \mathbb{R}$ the Sobolev space $H^s(M)$ can be characterized via pseudodifferential operators as

$$H^s(M) = \{u \in \mathscr{D}'(M) \mid Au \in L^2(M) \text{ for all } A \in \Psi^s(M)\}. \tag{6}$$

Since M is compact, we can cover M by a finite number of coordinate charts. It follows

Theorem 2.3. *Let M be a compact Riemannian manifold.*

1. *If $A \in \Psi^0(M)$, then $A : L^2(M) \to L^2(M)$ is bounded.*
2. *If $A \in \Psi^m(M)$ for $m < 0$, then $A : L^2(M) \to L^2(M)$ is compact.*
3. *For any $A \in \Psi^m(M)$ and any $s \in \mathbb{R}$, A maps $H^s(M)$ into $H^{s-m}(M)$.*

Remark. With a little bit more work, one can construct, for any elliptic pseudodifferential operator $P \in \Psi^m(M)$ a parametrix $Q \in \Psi^{-m}(M)$ so that $PQ = I + \Psi^{-\infty}(M)$ and $QP = I + \Psi^{-\infty}(M)$. Also one can prove some analogues of Gårding inequality and the Egorov theorem on compact manifolds.

¶ The principal symbol map.

Finally we associate with any pseudodifferential operator $A \in \Psi^m(M)$ a principal symbol $a \in S^m(T^*M)$.

Theorem 2.4. *There exists linear maps*

$$\sigma : \Psi^m(M) \to S^m(T^*M)/\hbar S^{m-1}(T^*M) \tag{7}$$

and a linear map [3]

$$Op : S^m(T^*M) \to \Psi^m(M) \tag{8}$$

such that

$$\sigma(Op(a)) = [a] \in S^m(T^*M)/\hbar S^{m-1}(T^*M).$$

Proof. Let $A \in \Psi^m(M)$ be a semiclassical pseudodifferential operator, namely

$$A = \sum_j M_{\chi_j} \varphi_j^* \widehat{a_j}^W (\varphi_j^{-1})^* M_{\chi_j} + \hbar^\infty \Psi^{-\infty}(M)$$

we define

$$\sigma_m(A)(x,\xi) := \sum_j \chi_(x)^2 \widetilde{\varphi_j}^* a_j \mod \hbar S^{m-1}(T^*M)$$

One can check that it is well-defined. In fact, According to (3), the leading term of the symbol is invariant under coordinate change, so that the above formula makes sense. Furthermore, according to Proposition 2.2, for any coordinate chart

[3]The map Op is non-canonical, for example it depends on the choice of P.O.U.

$(\varphi_\alpha, U_\alpha, V_\alpha)$ and any cut-off function $\chi \in C_0^\infty(U_\alpha)$, there exists a symbol $a_{\alpha,\chi} \in S^m$ such that

$$\chi(x)^2 \sigma_m(A) = \widetilde{\varphi_\alpha}^* a_{\alpha,\chi}.$$

So the value of $\sigma_m(A)$ is independent of the decomposition formula of A.

Conversely, for any $a \in S^m(T^*M)$, we choose a partition of unity χ_j^2 subordinate to a finite coordinate charts, namely

$$\text{supp}\chi_j \subset U_j, \quad \sum_{j=1}^K \chi_j^2 = 1$$

and define

$$A = \sum_j M_{\chi_j} \varphi_j^* \widetilde{a}^W (\varphi_j^{-1})^* M_{\chi_j},$$

then $A \in \Psi^m(M)$ with $\sigma(A) = a$. □

It is quite obvious that the principal symbol of pseudodifferential operators satisfies all the nice properties we listed last time for differential operators:

Proposition 2.5. *If $A \in \Psi^k(M), B \in \Psi^l(M)$, then*

1. $AB \in \Psi^{k+l}(M)$ *and* $\sigma_{k+l}(AB) = \sigma_k(A)\sigma_l(B)$.
2. $[A, B] \in \hbar\Psi^{k+l-1}(M)$ *and* $\sigma_{k+l-1}([A, B]) = \frac{\hbar}{i}\{\sigma_k(A), \sigma_l(B)\}$.
3. *If* $\sigma_k(A) = 0$, *then* $A \in \hbar\Psi^{k-1}(M)$.

Example. Any differential operator of order m is automatically a pseudodifferential operator of order m. So all examples we discussed last time are examples of pseudodifferential operators on manifolds.

Example. For a compact region U (with smooth boundary ∂U) in a Riemannian manifold (M, g), we can define the Dirichlet-to-Neumann map as in PSet 2. Then the Dirichlet-to-Neumann map is a pseudodifferential operator whose principal symbol is the function $\sigma(x, \xi) = |\xi|_x$ defined on $T^*(\partial U)$.