LECTURE 21: EIGENVALUES AND EIGENFUNCTIONS OF $\hbar$-PsDOs

In the next several lectures, we apply the theory of semiclassical pseudodifferential operators to study spectral theory of Schrodinger operators on $\mathbb{R}^n$ and on compact Riemannian manifolds (and in particular study the spectral theory of the Laplace-Beltrami operator on compact Riemannian manifolds). Here by "spectral theory" we means

(1) the asymptotic distribution of eigenvalues,
(2) the spacial "distribution" of eigenfunctions (in phase space[1]).

In particular we would like to prove Weyl law and the quantum ergodicity theorem that we mentioned in Lecture 1.

1. General spectral results of $\hbar$-pseudodifferential operators

¶Discrete spectrum.

As we have mentioned at the beginning of the course, we would like to study semiclassical pseudodifferential operators with discrete spectrum, in which case the the eigenvalues can be viewed as quantum energies (and the eigenfunctions can be viewed as quantum states whose energies are the corresponding eigenvalues) whose semiclassical behavior should be closely related to the behavior of the classical system. Moreover, we have mentioned at the beginning of Lecture 12 that a very useful way to prove the discreteness of spectrum is through compact operators, because for a compact operator A on a separable Hilbert space H,

- the spectrum $\sigma(A)$ consists of eigenvalues of finite multiplicities (with the only possible exception being the origin which could be an accumulation point of the spectrum),
- moreover if A is also self-adjoint, then the eigenvalues λ_k's are all real, and the corresponding eigenfunctions φ_k's can be taken to be an orthonormal basis of H, so that A can be written as

$$Au(x) = \sum_k \lambda_k \langle u, \varphi_k \rangle \varphi_k(x).$$

As a simple application of this idea, we prove

[1]although eigenfunctions are functions defined on the configuration space, it it still possible to study its "phase space distribution", as can be seen in today's lecture.

Theorem 1.1. *Suppose m is an order function on $\mathbb{R}^{2n}$ with*

$$\lim_{|(x,\xi)|\to\infty} m(x,\xi) = +\infty.$$

Suppose $p \in S(m)$ is real-valued and almost elliptic in $S(m)$(i.e. there exists $C > 0$ such that $p + C$ is elliptic). Then for $\hbar > 0$ small enough, the operator $\widehat{p}^W$ is an unbounded linear operator on $L^2(\mathbb{R}^n)$ with domain $H_\hbar(m)$, the eigenvalues of $\widehat{p}^W$ are discrete real numbers with finite multiplicities which diverges to ∞, and the eigenfunctions of $\widehat{p}^W$ can be chosen to form an orthonormal basis of $L^2(\mathbb{R}^n)$.

Proof. We use many results that we proved earlier:

- Proposition 2.1 in Lecture 16: $p \in S(m) \Longrightarrow \widehat{p}^W$ is well-defined as a map $\widehat{p}^W : H_\hbar(m) \to L^2(\mathbb{R}^n)$.
- Corollary 2.6 in Lecture 16: $p + C$ is elliptic in $S(m) \Longrightarrow A := \widehat{p+C}^W$: $H_\hbar(m) \to L^2(\mathbb{R}^n)$ has an inverse $B := \widehat{b}^W : L^2(\mathbb{R}^n) \to H_\hbar(m)$ with $b \in S(1/m)$.
- Theorem 1.5 in Lecture 12: $\lim_{|(x,\xi)|\to\infty} m(x,\xi) = +\infty \Longrightarrow \widehat{b}^W$ is a compact operator on $L^2(\mathbb{R}^n)$, whose eigenvalues has to be discrete (with finite multiplicity) with 0 as the only accumulation point.
- Computations at the beginning of Lecture 17: p is almost elliptic $\Longrightarrow p + i$ is elliptic.
- Corollary 2.2 in Lecture 16: $p + i$ is elliptic $\Longrightarrow \widehat{p}^W$, and thus $A = \widehat{p+C}^W$: $H_\hbar(m) \subset L^2 \to L^2$ is self-adjoint.
  A consequence: $B = \widehat{b}^W : L^2 \to L^2$ is also self-adjoint.
  To see this we start with any $u, v \in L^2(\mathbb{R}^n)$. Then there exists $u', v' \in H_\hbar(m)$ such that $\widehat{p+C}^W u' = u$ and $\widehat{p+C}^W v' = v$. Thus
  $$\langle Bu, v\rangle = \langle BAu', Av'\rangle = \langle u', Av'\rangle = \langle Au', v'\rangle = \langle u, Bv\rangle.$$
- Garding inequality in Lecture 15: The symbol of B is positive for $\hbar$ small enough (since the leading term is $1/(p + C) > 0) \Longrightarrow$ the eigenvalues of B are nonnegative.

Thus by the spectral theory of compact self-adjoint operators, we can write B as

$$Bu = \sum_k \widetilde{\lambda}_k \langle u, \varphi_k\rangle \varphi_k(x).$$

with $\widetilde{\lambda}_k \to 0^+$. It follows that $\widehat{p}^W$ has discrete spectrum (with finite multiplicities) $\frac{1}{\widetilde{\lambda}_k} - C$ which diverges to $+\infty$ (which also implies that $\widehat{p}^W$ is unbounded on $L^2(\mathbb{R}^n)$), and the eigenfunctions of $\widehat{p}^W$ (which are the same as eigenfunctions of the compact operator B) can be chosen to form an orthonormal basis of $L^2(\mathbb{R}^n)$. □

Remark. In the case of compact Riemannian manifold, it is easy to see that the same conclusion holds if we assume $p \in S^m(T^*M)$ for some $m > 0$.

Example. Consider Schrödinger operator

$$P = -\hbar^2\Delta + V$$

on $L^2(\mathbb{R}^n)$, where $V \in C^\infty(\mathbb{R}^n)$ is a real-valued smooth function such that[2]

(1) V has "polynomial growth" in the sense that there exists $k > 0$ such that

$$|\partial_x^\alpha V(x)| \le C_\alpha \langle x\rangle^k,$$

(2) V is "almost elliptic" in $S(\langle x\rangle^k)$ in the sense that

$$V(x) \ge c\langle x\rangle^k - C.$$

[Note that this implies $V(x) \to +\infty$ as $|x| \to \infty$. Of course you may replace $\langle x\rangle^k$ by any order function $\widetilde{m}(x)$ which diverges to $+\infty$ as $x \to \infty$.]

Then we can apply the Proposition 1.1 to $P + C \cdot \mathrm{Id}$ with order function

$$m(x,\xi) = \langle \xi\rangle^2 + \langle x\rangle^k$$

to conclude that P has discrete spectrum that diverges to ∞ and has an L^2-eigenbasis.

Example. Similarly, on a compact Riemannian manifold (M, g), for any potential function $V \in C^\infty(M)$, the Schrödinger operator

$$P = -\hbar^2\Delta + V$$

(and in particular the Laplace operator Δ itself) has discrete spectrum that diverges to ∞ and has an L^2-eigenbasis.

¶Regularity of eigenfunctions.

Although at first glance, the eigenfunctions of $\widehat{p}^W$ are only L^2-functions, they have much better regularity:

Theorem 1.2. *Suppose $m \ge 1$ is an order function on $\mathbb{R}^{2n}$, $p \in S(m)$ is a real-valued almost elliptic symbol. Suppose $u_\hbar \in L^2(\mathbb{R}^n)$ is an eigenfunction of $\widehat{p}^W$, i.e.*

$$\widehat{p}^W u_\hbar(x) = \lambda_\hbar u_\hbar(x). \tag{1}$$

Then for any $k = 0, 1, 2, \cdots$, we have $u_\hbar \in H_\hbar(m^k)$. Moreover, if $\lambda_\hbar \in [\alpha, \beta]$ for some constants α, β, then there exists constants C_k so that

$$\|u_\hbar\|_{H_\hbar(m^k)} \le C_k \|u_\hbar\|_{L^2}.$$

[2]According to a theorem of Friedrichs, if $V(x) \to +\infty$ as $|x| \to \infty$, then P has discrete spectrum. So the polynomial growth condition can be removed. For a proof, c.f. Theorem XIII.16 in *M. Reed and B. Simon, Methods of Modern Mathematical Physics, Volume 4.*

Proof. We have
$$(\widehat{p}^W + C)u_\hbar = (\lambda_\hbar + C)u_\hbar.$$
By ellipticity, there exists symbol $b \in S(1/m)$ so that $(\widehat{p}^W + C)^{-1} = \widehat{b}^W$. It follows that
$$(\widehat{b}^W)^k L^2 \subset H_\hbar(m^k).$$
Since
$$u_\hbar = (\lambda_\hbar + C)^k (\widehat{b}^W)^k u_\hbar,$$
the conclusion follows. □

Example. For the Schrodinger operator $H = -\hbar^2\Delta + V(x)$ on $\mathbb{R}^n$ whose potential is almost elliptic and of polynomial growth as described above, then the eigenfunction $u_\hbar \in \mathscr{S}(\mathbb{R}^n)$. [We have seen this for the Harmonic oscillator in Lecture 3.]

¶Concentration of eigenfunction in phase space.

Next we prove the following classical-quantum correspondence phenomena: the eigenfunctions associated to eigenvalues close to a given "energy level" E will "concentrate" on the corresponding energy surface in the phase space:

Theorem 1.3. *Suppose m is an order function satisfying $\lim_{|(x,\xi)|\to\infty} m(x,\xi) = +\infty$, $p \in S(m)$ is a real-valued almost elliptic symbol. Suppose $u_\hbar \in L^2(\mathbb{R}^n)$ is such that $\widehat{p}^W u = \lambda_\hbar u_\hbar$. Suppose $c \in S(1)$ is a symbol satisfying*
$$\{(x,\xi) \mid p(x,\xi) = E\} \cap \operatorname{supp}(c) = \emptyset,$$
where $E > 0$. Then there exists sufficiently small $\delta > 0$ such that if $|\lambda_\hbar - E| \le \delta$, then
$$\|\widehat{c}^W u_\hbar\|_{L^2(\mathbb{R}^n)} = O(\hbar^\infty)\|u_\hbar\|_{L^2(\mathbb{R}^n)}.$$

Proof. Since the order function m is proper and p is almost elliptic, p is also proper and thus the level set $p^{-1}(E) = \{(x,\xi) \mid p(x,\xi) = E\}$ is compact. So we can find a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that
$$0 \le \chi \le 1, \quad \chi \equiv 1 \text{ on } p^{-1}(E), \quad \chi \equiv 0 \text{ on } \operatorname{supp}(c).$$
Consider the symbol
$$b(x,\xi) = p(x,\xi) - \lambda_\hbar + i\chi$$
Then b is elliptic in $S(m)$. Thus
$$\widehat{b}^W = \widehat{p}^W - \lambda_\hbar + i\widehat{\chi}^W$$
is invertible with inverse $(\widehat{b}^W)^{-1} = \widehat{b_1}^W$ for some $b_1 \in S(1/m)$. Since $\operatorname{supp}(c) \cap \operatorname{supp}(\chi) = \emptyset$, we have (c.f. Corollary 1.4 in Lecture 9)
$$\widehat{c}^W \widehat{b_1}^W \widehat{\chi}^W = O(\hbar^\infty).$$
It follows
$$\widehat{c}^W u_\hbar = \widehat{c}^W \widehat{b_1}^W (\widehat{p}^W - \lambda_\hbar + i\widehat{\chi}^W)u_\hbar = O(\hbar^\infty).$$
□

2. Wavefront set properties

¶Wavefront set of a semiclassical family of states.

Now we introduce the conception of wavefront set associated with a family of wavefunctions $u_\hbar$ (not necessarily eigenfunctions). Roughly speaking, the wavefront set of a family of functions $u_\hbar$ is the region in the phase space where $u_\hbar$ are *microlocalized*:

Definition 2.1. Let $u_\hbar$ be a family of L^2-normalized functions on $\mathbb{R}^n$. We define the *semiclassical wavefront set* (which is also known as the *frequency set*, as it was first studied by V. Guillemin and S. Sternberg in their classical book *Geometric Asymptotics*) of $u_\hbar$ to be the set $\mathrm{WF}_\hbar(u) \subset T^*\mathbb{R}^n$ characterized by

$$(x_0, \xi_0) \notin \mathrm{WF}_\hbar(u) \Longleftrightarrow \exists a \in S(1) \text{ with } |a(x_0,\xi_0)| \ge c > 0 \text{ for all } \hbar, \text{ such that } \|\widehat{a}^W u_\hbar\|_{L^2} = O(\hbar^\infty). \tag{2}$$

By definition, the complement of $\mathrm{WF}_\hbar(u)$ is always open, thus $\mathrm{WF}_\hbar(u)$ must be a closed subset of $\mathbb{R}^{2n}$.

Example (truncated plane wave). Fix $\xi_0 \in \mathbb{R}^n$ and $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\|\chi\|_{L^2} = 1$. Consider the following family of L^2-normalized fast oscillating functions

$$e_{\xi_0}(x,\hbar) := \chi(x) e^{i\xi_0\cdot x/\hbar}.$$

Then

$$\mathrm{WF}_\hbar(e_{\xi_0}) = \{(x,\xi_0) \mid x \in \mathrm{supp}(\chi)\}.$$

To see this we calculate

$$\widehat{a}^W u(x) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(x-y)\cdot\xi} a(x,\xi)\chi(y) e^{\frac{i}{\hbar}\xi_0\cdot y} dy d\xi$$

The critical point of the phase function $\varphi_x(y,\xi) := (x-y)\cdot\xi + \xi_0 \cdot y$ is

$$y = x, \quad \xi = \xi_0.$$

So for any $(x_1,\xi_1) \notin \{(x,\xi_0) \mid x \in \mathrm{supp}(\chi)\}$, if we can take a such that $a(x_1,\xi_1) \neq 0$ and $(x,\xi_0) \notin \mathrm{supp}(a)$, then by "non-stationary phase lemma" (Proposition 1.2 in Lecture 5) we get $\widehat{a}^W u(x) = O(\hbar^\infty)$ and thus $(x_1,\xi_1) \notin WF_\hbar(e_{\xi_0})$.

Conversely suppose x is a point with $\chi(x) > 0$ and suppose $a(x,\xi_0) > 0$. Then by the lemma of stationary phase,

$$\widehat{a}^W u(x) = Ca(x,\xi_0)\chi(x) + O(\hbar)$$

for some nonzero constant C. So $(x,\xi_0) \in \mathrm{WF}_\hbar(e_{\xi_0})$. Since $\mathrm{WF}_\hbar(e_{\xi_0})$ is closed, the conclusion follows.

It turns out that in the definition of wavefront set, we may replace $a \in S(1)$ by $a \in C_0^\infty(\mathbb{R}^{2n})$:

Proposition 2.2. *Suppose $(x_0, \xi_0) \notin \mathrm{WF}_\hbar(u)$, then for any $b \in C_0^\infty(\mathbb{R}^{2n})$ with support sufficiently close to (x_0, ξ_0), we have $\|\widehat{b}^W u_\hbar\|_{L^2} = O(\hbar^\infty)$.*

Proof. Suppose $a \in S(1)$ is a symbol satisfying (2). Take a cut-off $\chi \in C_0^\infty(\mathbb{R}^{2n})$ with $\chi \equiv 1$ in a neighborhood of $\mathrm{supp}(b)$, such that

$$|\chi(x,\xi)(a(x,\xi) - a(x_0,\xi_0)) + a(x_0,\xi_0)| \geq c/2 > 0,$$

i.e. it is elliptic in $S(1)$ (here we used "$\mathrm{supp}(b)$ is sufficiently close to (x_0,ξ_0)"). So by Theorem 1.7 in Lecture 14, for $\hbar$ small enough, there exists $c \in S(1)$ such that

$$\widehat{c}^W = [\widehat{\chi}^W(\widehat{a}^W - a(x_0,\xi_0)\mathrm{Id}) + a(x_0,\xi_0)\mathrm{Id}]^{-1}.$$

So we have

$$\widehat{b}^W = \widehat{b}^W \widehat{c}^W \widehat{\chi}^W \widehat{a}^W + a(x_0,\xi_0)\widehat{b}^W \widehat{c}^W \widehat{1-\chi}^W.$$

By condition (2), the first term is $O(\hbar^\infty)$. Since $\mathrm{supp}(b) \cap \mathrm{supp}(1-\chi) = \emptyset$, Corollary 1.4 in Lecture 9 tells us that the second term is $O(\hbar^\infty)$. □

Wavefront set has the following remarkable property:

Theorem 2.3. *Suppose $a = a(x,\xi;\hbar) \in S(m)$ for all $\hbar$. Then*

$$\mathrm{WF}_\hbar(\widehat{a}^W u) \subset \mathrm{WF}_\hbar(u).$$

Proof. Suppose $(x_0,\xi_0) \notin \mathrm{WF}_\hbar(u)$. Choose $b \in C_0^\infty(\mathbb{R}^{2n})$ such that $b(x_0,\xi_0) \neq 0$ and $\widehat{b}^W u = O(\hbar^\infty)$. Then

$$\widehat{b}^W \widehat{a}^W = \widehat{b \star a}^W = \widehat{c}^W + O(\hbar^\infty)$$

for some symbol $c \in C_0^\infty(\mathbb{R}^{2n})$ satisfying $\mathrm{supp}(c) \subset \mathrm{supp}(b)$. It follows

$$\|\widehat{b}^W \widehat{a}^W u\|_{L^2} = \|\widehat{c}^W u\|_{L^2} + O(\hbar^\infty) = O(\hbar^\infty)$$

which implies $(x_0,\xi_0) \notin \mathrm{WF}_\hbar(\widehat{a}^W u)$. □

As a consequence, we immediately get the following nice property of elliptic operators:

Corollary 2.4. *If a is elliptic in $S(m)$, then* $\mathrm{WF}_\hbar(\widehat{a}^W u) = \mathrm{WF}_\hbar(u)$

Remark. Wavefront set is invariant under coordinate change: If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism that is identity outside a compact set, then

$$\mathrm{WF}_\hbar(f^* u) = \{(x, (df)^T \xi) : (f(x), \xi) \in \mathrm{WF}_\hbar(u)\}.$$

As a consequence, it is well-defined as a subset in T^*M for compact manifolds M.

¶Wavefront set of a semiclassical family of eigenfunctions.

For eigenfunctions, we have

Theorem 2.5. *Suppose $\lambda_\hbar$ are eigenvalues of $\widehat{p}^W$ such that $|\lambda_\hbar - E_0| < \varepsilon$.*[3] *Let $u_\hbar$ be the associated eigenfunctions. Then the wavefront set $\mathrm{WF}_\hbar(u_\hbar)$ of $u_\hbar$ is a non-empty subset contained in $p^{-1}(E_0)$.*

Proof. The fact $\mathrm{WF}_\hbar(u_\hbar) \subset p^{-1}(E_0)$ follows from Theorem 1.3. It remains to prove $\mathrm{WF}_\hbar(u_\hbar) \neq \emptyset$. Suppose to the contrary that $\mathrm{WF}_\hbar(u_\hbar) = \emptyset$, i.e. for any $(x_0, \xi_0) \in p^{-1}(E_0)$, there exists $a_{x_0,\xi_0} \in C_0^\infty(\mathbb{R}^{2n})$ with $a_{x_0,\xi_0}(x_0, \xi_0) = 1$ such that

$$\widehat{a_{x_0,\xi_0}}^W u_\hbar = O(\hbar^\infty).$$

We cover the compact set $p^{-1}(E)$ by finitely many balls $B((x_k, \xi_k), r_k)$, such that $(x_k, \xi_k) \in p^{-1}(E)$ and $a_{x_k,\xi_k}(x, \xi) > 1/2$ for $(x, \xi) \in B((x_k, \xi_k), r_k)$. We define $\widetilde{a} = \sum a_{x_k,\xi_k}$. Then $\widetilde{a} > 1/2$ in a neighborhood of $p^{-1}(E_0)$. Replace a_{x_k,ξ_k} by $a_{x_k,\xi_k}/\widetilde{a}$ and let $a =$. Then $a \equiv 1$ in a neighborhood of $p^{-1}(E_0)$. By Theorem 1.3,

$$\widehat{a}^W u_\hbar = u_\hbar + \widehat{1-a}^W u_\hbar = u_\hbar + O(\hbar^\infty),$$

a contradiction. □

[3] According to Weyl's law that we will prove later, as $\hbar \to 0$ there are many eigenvalues satisfying this condition.