LECTURE 22-23: WEYL'S LAW

1. Functional calculus of pseudodifferential operators

In sections 1 and 2 we will always assume

- $m \geq 1$ is an order function,
- $p \in S(m)$ is a real-valued symbol,
- $p + i$ is elliptic in $S(m)$.

Under these assumptions we know that

$$P = \widehat{p}^W : H_\hbar(m) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is a (densely-defined) self-adjoint operator, and moreover, $P \pm i \cdot \mathrm{Id}$ is invertible for $\hbar \in (0, \hbar_0)$ and the inverse is a pseudodifferential operator with symbol in $S(1/m)$.

¶Helffer-Sjöstrand formula.

In Lecture 8 we mentioned that for a self-adjoint operator P on a Hilbert space H and a Borel measurable function f on $\mathbb{R}$, one can define a new self-adjoint linear operator $f(P)$ on H using the spectral theorem as follows: By spectral theorem (multiplication form) there is a measurable space (X, μ), a measurable real-valued function h on X and a unitary isomorphism $V : H \to L^2(X, \mu)$ so that

$$V \circ P \circ V^* = M_h$$

on $L^2(X, \mu)$. Then the operator $f(P)$ is defined to be

$$f(P) = V^* \circ M_{f(h(x))} \circ V.$$

We notice that by definition,

$$\text{If } |f| \leq C, \text{ then } \|f(P)\|_{\mathcal{L}} \leq C. \tag{1}$$

We want to answer the following natural question:

> Question: Is $f(P)$ a semiclassical pseudodifferential operator if P is a semiclassical pseudodifferential operator and f is a (nice) function?

Unfortunately the construction of $f(P)$ above is a bit too abstract to work with. However, if $f \in \mathscr{S}$ is a Schwartz function, then by using the so-called *almost analytic extension* $\tilde{f}$ of f, Helffer-Sjöstrand gave a more concrete formula for $f(P)$, namely

$$f(P) = -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z)(z - P)^{-1} L(dz) \tag{2}$$

using which we will prove that $f(P)$ is a semiclassical pseudodifferential operator, and calculate its symbol expansion. Recall that an almost analytic extension $\tilde{f} \in$

$C^\infty(\mathbb{C})$ of a Schwartz function $f \in \mathscr{S}(\mathbb{R})$ is by definition a smooth function on $\mathbb{C}$ such that
$$\tilde{f}|_{\mathbb{R}} = f, \quad \text{supp}\tilde{f} \subset \{z : |\text{Im}(z)| \le 1\}$$
and such that as $|\text{Im}(z)| \to 0$,
$$\bar{\partial}_z \tilde{f}(z) = O(|\text{Im}(z)|^\infty),$$
where as usual, $\bar{\partial}_z = (\partial_x + i\partial_y)/2$ for $z = x + iy$, and $L(dz)$ denotes the Lebesgue measure on $\mathbb{C}$ (we don't use $dxdy$ since x has different meaning below.) To prove formula (2) it is enough to notice the following relation between f and $\tilde{f}$ (c.f. PSet 1):
$$f(t) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z)(z-t)^{-1} L(dz)$$
and use the spectral theorem (multiplication form) for both $(z-P)^{-1}$ and $f(P)$.

In literature there are at least two different ways to construct an almost analytic extension: the first construction is due to Hörmander (who proposed the conception of almost analytic extension in 1968) who adapted the construction in Borel's Lemma by putting
$$\tilde{f}(x+iy) = \sum_k \frac{f^{(k)}(x)}{k!}(iy)^k \chi(\lambda_k y),$$
where λ_k is a sequence of real numbers that is chosen so that they tends to $+\infty$ sufficiently fast, and χ is a cut-off function. The other way is due to Mather who make use of the Fourier transform (PSet 1):
$$\tilde{f}(x+iy) := \frac{1}{2\pi}\chi(y)\int_{\mathbb{R}} \chi(y\xi)\hat{f}(\xi)e^{i\xi(x+iy)}d\xi.$$
Of course the almost analytic extension is not unique in general. Also note that if f is compactly supported on $\mathbb{R}$, then we can take $\tilde{f}$ to be compactly-supported in $\mathbb{C}$ [For the first construction this is obvious, for the second construction we may multiply the formula by a cut-off function which is identically one on $\text{supp}(f)$].

¶Symbol of the resolvent $(z-P)^{-1}$.

We want to prove that $f(P)$ is a pseudodifferential operator if f is Schwartz function on $\mathbb{R}$ and P is a semiclassical pseudodifferential operator, and calculate its symbol. In view of the Helffer-Sjöstrant formula (2), we start with the resolvent operator $(z-P)^{-1}$.

Lemma 1.1. *Under the previous assumptions, for any $z \in \mathbb{C}\setminus\mathbb{R}$, the operator $z-P$ is invertible, and there exists $r_z \in S(1/m)$ such that $\widehat{r_z}^W = (z-P)^{-1}$.*

Proof. This is just a consequence of ellipticity: for any fixed $z = a + bi$ $(b \neq 0)$, we have
$$\inf_{t\in\mathbb{R}} \frac{|z-t|}{|t+i|} > 0.$$

We need a more explicit lower bound below. So let's try to find a constant $C > 0$ such that

$$\frac{|z-t|}{|t+i|} \geq bC, \quad \forall t \in \mathbb{R}. \tag{3}$$

This is equivalent to

$$(a-t)^2 + b^2 - (bC)^2(t^2+1) \geq 0, \quad \forall t \in \mathbb{R}.$$

We will take C small enough so that $bC < 1$. By calculating the discriminant and simplifying it, we get the condition on C:

$$\frac{1-C^2}{C^2}(1-C^2b^2) \geq a^2.$$

As a consequence, if we assume $|z| < C_0$, then we can find a constant C so that (3) holds for all $t \in \mathbb{R}$.

It follows that

$$|z-p| \geq C|p+i| \geq cm$$

and thus $z-p$ is elliptic in $S(m)$. □

To apply Helffer-Sjöstrand formula, we need to study the dependence of r_z on z. For this purpose, we introduce the following variation of Beals's theorem.

Recall that in Lecture 14 we have seen that any continuous linear operator $A : \mathscr{S} \to \mathscr{S}'$ can be written as $A = \widehat{a}^W$ for some $a = a(x,\xi,\hbar) \in \mathscr{S}'$, and Beals's theorem tells us $a \in S(1)$ if and only if

$$\|\mathrm{ad}_{\widehat{l_1}^W} \circ \cdots \circ \mathrm{ad}_{\widehat{l_N}^W} A\|_{\mathcal{L}(L^2)} = O(\hbar^N)$$

for any N and any linear functions $l_1, \cdots, l_N$ on $\mathbb{R}^{2n}$. Of course the main part in the proof of Beals's theorem is to prove the condition above implies $a \in S(1)$, namely, to prove $|\partial_{x,\xi}^\alpha a| \leq C_\alpha$.

Proposition 1.2 (Beals's Estimate with Parameter)**.** *Suppose $a = a(x,\xi,z;\hbar)$, i.e. a depends on a parameter z. Let $\delta = \delta(z)$ be a function valued in $(0,1]$ such that*

$$\|\mathrm{ad}_{\widehat{l_1}^W} \circ \cdots \circ \mathrm{ad}_{\widehat{l_N}^W} A\|_{\mathcal{L}(L^2)} = O(\delta^{-N}\hbar^N)$$

holds for any N and any linear functions $l_1, \cdots, l_N$ on $\mathbb{R}^{2n}$. Then there exists a universal constant M such that for any α,

$$|\partial_{x,\xi}^\alpha a(x,\xi,z;\hbar)| \leq C_\alpha \max(1, \sqrt{\hbar}/\delta)^M \delta^{-|\alpha|}.$$

We will leave the proof as an exercise. [c.f. Helffer-Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit*, Prop. 8.4.]

Using this result, we will prove the following resolvent symbol estimate:

Theorem 1.3. *Fix $C_0 > 0$. For any $z \in \mathbb{C} \setminus \mathbb{R}$ with $|z| < C_0$, we have*

$$|\partial^\alpha_{x,\xi} r_z| \le C_\alpha \max(1, \hbar^{1/2}|\mathrm{Im}(z)|^{-1})^M |\mathrm{Im}(z)|^{-1-|\alpha|}.$$

where $r_z \in S(1/m)$ is the symbol of the resolvent of P, namely $(z-P)^{-1} = \widehat{r_z}^W$.

Proof. By Proposition 1.2, we only need to prove

$$\|\mathrm{ad}_{\widehat{l_1}^W} \circ \cdots \circ \mathrm{ad}_{\widehat{l_N}^W}(z-P)^{-1}\|_{\mathcal{L}(L^2)} = O(|\mathrm{Im}(z)|^{-1-N}\hbar^N). \tag{4}$$

Using the formulae

$$\mathrm{ad}_A(B^{-1}) = -B^{-1}(\mathrm{ad}_A B)B^{-1}$$

and

$$\mathrm{ad}_A(BC) = (\mathrm{ad}_A B)C + B\mathrm{ad}_A C$$

we can write $\mathrm{ad}_{\widehat{l_1}^W} \circ \cdots \circ \mathrm{ad}_{\widehat{l_N}^W}(z-P)^{-1}$ as a summation of terms of the form

$$\pm(z-P)^{-1}\mathrm{ad}^{\alpha_1}_{\widehat{l}^W}(P)(z-P)^{-1}\mathrm{ad}^{\alpha_2}_{\widehat{l}^W}(P)\cdots(z-P)^{-1}\mathrm{ad}^{\alpha_k}_{\widehat{l}^W}(P)(z-P)^{-1},$$

where $\alpha_j = \{\alpha_{j,1}, \cdots, \alpha_{j,n_j}\}$ such that $\{\alpha_{j,l} \mid \forall j, l\} = \{1, \cdots, N\}$. Note that

$$p \in S(m) \Longrightarrow \{l, p\} \in S(m) \Longrightarrow \mathrm{ad}^{\alpha_j}_{\widehat{l}^W}(P) = \widehat{p_j}^W \text{ for some } p_j \in \hbar^{n_j}S(m).$$

Thus in view of the fact $P+i$ has symbol in $S(1/m)$, we get that

$$\begin{aligned}\|\mathrm{ad}^{\alpha_j}_{\widehat{l}^W}(P)(z-P)^{-1}\|_{\mathcal{L}(L^2)} &\le \|\mathrm{ad}^{\alpha_j}_{\widehat{l}^W}(P)(P+i)^{-1}\|_{\mathcal{L}(L^2)} \cdot \|(P+i)(z-P)^{-1}\|_{\mathcal{L}(L^2)} \\ &\le O(\hbar^{n_j}|\mathrm{Im}(z)|^{-1}),\end{aligned}$$

where in the last step we used Calderon-Vaillancourt theorem for the first term, and

$$\|(P+i)(z-P)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C|\mathrm{Im}(z)|^{-1}$$

for the second term, which is a consequence of

- The fact (1) at the beginning of this lecture, which is a consequence of the spectral theorem,
- The argument in the proof of Lemma 1.1, i.e. if we assume $|z| < C_0$, then there exists a universal constant C that is independent of z such that (3) holds. In other words,

$$\frac{|z-t|}{|t+i|} \le C|\mathrm{Im}(z)|^{-1}, \quad \forall t \in \mathbb{R}.$$

Similarly we have

$$\|(z-P)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le |\mathrm{Im}(z)|^{-1},$$

so the estimate (4) holds, which completes the proof. □

¶The functional calculus.

Now we are ready to prove that the operator $f(P)$ is also a semiclassical pseudodifferential operator:

Theorem 1.4. *If $f \in \mathscr{S}$, then $f(P) = \widehat{a}^W$, where $a \in S(m^{-k})$ for any $k \in \mathbb{N}$. Moreover, we have an asymptotic expansion*

$$a(x,\xi) \sim \sum_{k\geq 0} \hbar^k a_k(x,\xi),$$

where $a_0(x,\xi) = f(p(x,\xi)$, and in general,

$$a_k(x,\xi) = \frac{1}{(2k)!} (\partial_t)^{2k} \left[f(t) q_k(x,\xi,t)\right]\Big|_{t=p(x,\xi)}.$$

Proof. Using Helffer-Sjöstrand formula, we see $f(P) = \widehat{a}^W$, where

$$a(x,\xi) = -\frac{1}{\pi}\int_{\mathbb{C}} (\bar{\partial}_z \tilde{f}(z)) r_z(x,\xi) L(dz),$$

where $\tilde{f}$ is an almost analytic extension of f. Although the (x,ξ)-derivatives of r_z is unbounded as $\mathrm{Im}(z) \to 0$, the unboundedness is controlled by Theorem 1.3. So by using the fact $\bar{\partial}_z \tilde{f}(z) = O(|\mathrm{Im}(z)|^\infty)$ we see $a \in S(1)$. More generally, for any $k \in \mathbb{N}$, if we apply the above arguments to $f_k(t) = f(t)(t+i)^k$, we can prove that $f_k(P) = (P+i)^k f(P)$ is a pseudodifferential operator with symbol in $S(1)$, which implies $a \in S(m^{-k})$.

We also need an asymptotic expansion of a. For this purpose we start with the asymptotic expansion of r_z. Recall that by construction (Lecture 14), $r_z = \widetilde{r_z} \star (1-u)$ for some $u \in \hbar^\infty S(1)$, and $\widetilde{r_z}$ can be solved form the equation

$$(z-p)\star \widetilde{r_z} - 1 = O(\hbar^\infty)$$

inductively, which has the form (exercise)

$$r_z \sim \sum_{k=0}^{\infty} \hbar^k \frac{q_k(x,\xi,z)}{(z-p(x,\xi))^{2k+1}} \tag{5}$$

where q_k is a degree $2k$ polynomial in z (and thus is holomorphic in z):

$$q_k(x,\xi,z) = \sum_{j=0}^{2k} q_{k,j}(x,\xi) z^j$$

with $q_0 = 1, q_1 = 0$ and in general, $q_{k,j} \in S(m^{2k-j})$.

Again the expansion (5) is an expansion for each fixed z, and is not a good one as $\mathrm{Im}(z) \to 0$. However, we may resolve this problem by fixing a $\delta \in (0, 1/2)$

and considering the two region $|\mathrm{Im}(z)| \le \hbar^\delta$ and $|\mathrm{Im}(z)| \ge \hbar^\delta$ separately. Since $\bar\partial_z \tilde f(z) = O(|\mathrm{Im}(z)|^\infty)$, we see

$$-\frac{1}{\pi}\int_{|\mathrm{Im}(z)|<\hbar^\delta} \bar\partial_z \tilde f(z) r_z(x,\xi)dxdy \in \hbar^\infty S(1/m).$$

For $|\mathrm{Im}(z)| > \hbar^\delta$, Theorem 1.3 implies that $r_z \in \hbar^\delta S_\delta(1/m)$. So the expansion (5) is an expansion in $\hbar^\delta S_\delta(1/m)$ in this case, and we thus we get from Hellfer-Sjöstrand formula the asymptotic expansion

$$a(x,\xi) \sim \sum_{k\ge 0} \hbar^k \tilde a_k(x,\xi)$$

in $\hbar^\delta S_\delta(1/m)$, where

$$\tilde a_k(x,\xi) = -\frac{1}{\pi}\int_{|\mathrm{Im}(z)|>\hbar^\delta} \bar\partial_z \tilde f(z) \frac{q_k(x,\xi,z)}{(z-p(x,\xi))^{2k+1}} L(dz).$$

Modulo $\hbar^\infty S(1/m)$, we may replace $\tilde a_k$ by

$$\begin{aligned} a_k(x,\xi) &= -\frac{1}{\pi}\int_{\mathbb{C}} \bar\partial_z \tilde f(z) \frac{q_k(x,\xi,z)}{(z-p(x,\xi))^{2k+1}} L(dz) \\ &= -\frac{1}{\pi}\frac{1}{(2k)!}\int_{\mathbb{C}} \bar\partial_z \left(\tilde f(z) q_k(x,\xi,z)\right) (-\partial_z)^{2k} \frac{1}{(z-p(x,\xi))} L(dz) \\ &= -\frac{1}{\pi}\frac{1}{(2k)!}\int_{\mathbb{C}} \bar\partial_z (\partial_z)^{2k} \left(\tilde f(z) q_k(x,\xi,z)\right) \frac{1}{(z-p(x,\xi))} L(dz) \\ &= \frac{1}{(2k)!} (\partial_t)^{2k} \left(f(t) q_k(x,\xi,t)\right)\Big|_{t=p(x,\xi)}, \end{aligned}$$

where we used the fact that q_k is a polynomial and thus is analytic in z, and the fact $\tilde f(z) q_k(x,\xi,z)$ is an almost analytic extension of $f(t)q_k(x,\xi,t)$. In particular,

$$a_0(x,\xi) = f(p(x,\xi)).$$

□

Remark. Suppose $p \sim p_1 + \hbar p_2 + \cdots$, then using $q_1 = 0$ we easily get

$$a \sim f(p_1) + \hbar f'(p_1) p_2 + \cdots .$$

2. Weyls' law for $\hbar$-Pseudodifferential Operators

¶A trace formula.

Now we are ready to prove

Theorem 2.1. *Suppose $I = (a, b)$ is a finite interval and suppose*

$$\liminf_{(x,\xi)\to\infty} p(x,\xi) > b.$$

Then for any $f \in C_0^\infty(I)$, the operator $f(P)$ is a trace class operator on $L^2(\mathbb{R}^n)$ with

$$\mathrm{tr} f(P) \sim (2\pi\hbar)^{-n} \sum_{k=0}^{\infty} \hbar^k \int_{\mathbb{R}^{2n}} a_j(x,\xi) dx d\xi, \tag{6}$$

where the leading term $a_0(x,\xi) = f(p(x,\xi))$.

Proof. Let $p_1 \in S(m)$ be a real-valued symbol such that

$$p - p_1 \in C_c^\infty(\mathbb{R}^{2n}) \quad \text{and} \quad \inf p_1 > b.$$

Then $p_1 + i$ is also elliptic in $S(m)$. As a consequence, both $P = \widehat{p}^W$ and $P_1 = \widehat{p_1}^W$ are densely defined self-adjoint operator mapping $H_\hbar(m) \subset L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Moreover, there is an open neighborhood Ω of $\overline{I} = [a, b]$ such that $(z - P_1)^{-1}$ is holomorphic for z in Ω.

For any $f \in C_0^\infty(I)$, we let $\widetilde{f}$ be an almost holomorphic extension of f such that $\mathrm{supp}(\widetilde{f}) \subset \Omega$. Since $\mathrm{Spec}(P_1) \cap I = \emptyset$, we have $\widetilde{f}(P_1) = 0$.[1]

For $\mathrm{Im} z \neq 0$, from $z - P_1 = z - P + P - P_1$ we get the following resolvent identity

$$(z - P)^{-1} = (z - P_1)^{-1} + (z - P)^{-1}(P - P_1)(z - P_1)^{-1}.$$

It follows from the Hellfer-Sjöstrand formula that

$$f(P) = -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z) [(z - P)^{-1}(P - P_1)(z - P_1)^{-1}] L(dz).$$

Since $p - p_1$ is compactly supported, the operator $P - P_1$ is trace class. It follows that $f(P)$ has finite trace norm and thus is also trace class. It follows from Lecture 13 that

$$\mathrm{tr} f(P) = \frac{1}{(2\pi\hbar)^n} \int a(x,\xi) dx d\xi.$$

Now the conclusion follows. □

[1] This is also a consequence of Hellfer-Sjöstrand formula.

¶Weyl's law.

Denote

$$N_\hbar(P,[a,b]) = \#(\mathrm{Spec}(P)\cap[a,b])$$

be the number of eigenvalues of $P_\hbar$ in the interval $[a,b]$. To estimate $N_\hbar(P,[a,b])$, we approximate the characteristic function of the interval $[a,b]$ by smooth functions both from below and from above. Thus it is natural to introduce

$$\underline{V}([a,b]) = \lim_{\varepsilon\to 0+} \mathrm{Vol}(p^{-1}([a+\varepsilon, b-\varepsilon]))$$

and

$$\overline{V}([a,b]) = \lim_{\varepsilon\to 0+} \mathrm{Vol}(p^{-1}([a-\varepsilon, b+\varepsilon])).$$

As a direct consequence of the trace formula, we get

Theorem 2.2 (Weyl's law)**.** *For any $a<b$, as $\hbar\to 0$ we have*

$$\frac{1}{(2\pi\hbar)^n}(\underline{V}([a,b])+o(1)) \le N_\hbar(P,[a,b]) \le \frac{1}{(2\pi\hbar)^n}(\overline{V}([a,b])+o(1)). \tag{7}$$

Proof. Pick two sequence of compactly supported smooth functions $\underline{f_\varepsilon}, \bar{f_\varepsilon}$ (e.g. by regularization via convolution) that approaches the characteristic function $\chi_{[a,b]}$ of the interval $[a,b]$ from below and from above, namely

$$1_{[a+\varepsilon,b-\varepsilon]} \le \underline{f_\varepsilon} \le 1_{[a,b]} \le \bar{f_\varepsilon} \le 1_{[a-\varepsilon,b+\varepsilon]}.$$

Then we have

$$\mathrm{tr}\underline{f_\varepsilon}(P) \le N_\hbar(P,[a,b]) \le \mathrm{tr}\bar{f_\varepsilon}(P).$$

So the conclusion follows from the trace formula we just proved.

□

In particular, for Schrödinger operator $P=-\hbar^2\Delta+V$, where V satisfying the "polynomial growth" and "almost elliptic" conditions that we mentioned last time, we have the following Weyl's law for Schrödinger operators:

Theorem 2.3 (Weyl's law for Schrödinger operator on $\mathbb{R}^n$)**.** *For any $a<b$,*

$$N_\hbar(P,[a,b]) = \frac{1}{(2\pi\hbar)^n}\left(\mathrm{Vol}\{(x,\xi)\mid a\le|\xi|^2+V(x)\le b\}+o(1)\right). \tag{8}$$

Note that a special case of this theorem (namely P is the harmonic oscillator) has been proven in Lecture 3.

3. Weyl's law for $\Psi(M)$

In this section we always assume

- (M, g) is a compact Riemannian manifold,
- $m > 0$ is a positive integer,
- $P : H_\hbar^m(M) \to L^2(M)$ is a self-adjoint pseudodifferential operator in $\Psi^m(M)$,
- the principal symbol $p = \sigma_m(P)$ is real-valued and almost elliptic in $S^m(T^*M)$. Moreover, for any $a \in \mathbb{R}$, $\lim_{\varepsilon\to 0} \mathrm{Vol}[p^{-1}(a-\varepsilon, a+\varepsilon)] = 0$.

¶Basic properties of eigenvalues/eigenfunctions.

By adapting the proofs of Theorem 1.1 and Proposition 1.2 in Lecture 21 to the setting of compact Riemannian manifolds, we have

Proposition 3.1. *Under the above assumptions,*

(1) *P has discrete real spectrum*
$$\mathrm{Spec}(P) : \lambda_1(\hbar) \le \lambda_2(\hbar) \le \cdots \le \lambda_n(\hbar) \le \cdots \to \infty.$$
(2) *Each eigenfunction $\varphi_j(x)$ is a smooth function, and $\{\varphi_j(x)\}$ can be taken to be an L^2-orthonormal basis.*

¶The functional calculus.

As we have seen, to prove Weyl's law, the crucial ingredient is the following

Theorem 3.2. *Suppose $f \in \mathscr{S}(\mathbb{R})$. Then $f(P) \in \Psi^{-\infty}(M)$ with principal symbol* [2]
$$\sigma(f(P)) = f(p(x,\xi)).$$

Proof. Idea: We first prove $f(P) \in \Psi^0(M)$. According to Proposition 2.2 in Lecture 20, it is enough to prove

(a) For any coordinate patch $(\varphi_\alpha, U_\alpha, V_\alpha)$, there exists $\chi \in C_0^\infty(U_\alpha)$ such that $(\varphi_\alpha^{-1})^* M_\chi f(P) M_\chi (\varphi_\alpha)^* \in \Psi^0(\mathbb{R}^n)$.
(b) For any $\chi_1, \chi_2 \in C^\infty(M)$ with $\mathrm{supp}(\chi_1) \cap \mathrm{supp}(\chi_2) = \emptyset$, we want to prove $M_{\chi_1} f(P) M_{\chi_2} \in \hbar^\infty \Psi^{-\infty}(M)$.

The passing from $\Psi^0(M)$ to $\Psi^{-\infty}$ is standard: one only need to apply the previous result to $(P+i)^k f(P) = g_k(P)$, where $g(t) = (t+i)^k f(t) \in \mathscr{S}$. Finally we calculate the principal symbol of $f(P)$ via the Helffer-Sjöstrand formula.

Step 1. We first prove (b), namely for any $\chi_1, \chi_2 \in C^\infty(M)$ with $\mathrm{supp}(\chi_1) \cap \mathrm{supp}(\chi_2) = \emptyset$, we want to prove $M_{\chi_1} f(P) M_{\chi_2} \in \hbar^N \Psi^{-N}(M)$ for any N. According to Beals's theorem, it is enough to prove
$$\|M_{\chi_1} f(P) M_{\chi_2}\|_{\mathcal{L}(H_\hbar^{-N}, H_\hbar^N)} = O(\hbar^N).$$

[2] Note that in our definition, $\Psi^{-\infty}(M)$ is not negligible, and the principal symbol of an element in $\Psi^{-\infty}(M)$ is an element in $S^{-\infty}(T^*M)$. Only elements in $\hbar^\infty \Psi^{-\infty}(M)$ are negligible and has zero principal symbol of any order.

According to the Hellfer-Sjöstrand formula

$$f(P) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z)(z-P)^{-1}\mathcal{L}(dz),$$

it is enough to prove

$$\|M_{\chi_1}(z-P)^{-1}M_{\chi_2}\|_{\mathcal{L}(H_\hbar^{-N},H_\hbar^N)} = O(\hbar^N|\mathrm{Im}(z)|^{-K_N}) \tag{9}$$

for some $K_N > 0$. [We can't conclude (9) directly since we don't know where the resolvent $(z-P)^{-1}$ is a semiclassical pseudodifferential operator or not. So the idea is: approximate $(z-P)^{-1}$ by a semiclassical pseudodifferential operator so that we can control both the norm of the expression (9) with $(z-P)^{-1}$ replaced by the semiclassical pseudodifferential operator, and the norm of the remainder.]

For $z \in \mathbb{C}\setminus\mathbb{R}$, we let $Q_0(z) = \mathrm{Op}((z-p)^{-1}) \in \Psi^{-m}(M)$. According to Proposition 2.5 in Lecture 20,

$$\sigma_0(\mathrm{Id} - (z-P)Q_0) = 1 - \sigma_m(z-P)\sigma_{-m}(Q_0) = 0$$

and thus

$$(z-P)Q_0 = \mathrm{Id} - R_1$$

for some $R_1(z) \in \hbar\Psi^{-1}(M)$. According to the Calderon-Vailancourt theorem, the operator norm of $\|R_1(z)\|_{\mathcal{L}(H_\hbar^{-N},H_\hbar^{-N+1})}$ is controlled by finitely many derivatives of $r_1(z)$, which, by using the Moyal product formula in local charts, together with the resolvent estimate, namely Theorem 1.3, is controlled by $\mathrm{Im}(z)^{-K_1}$ for some $K_1 > 0$:

$$\|R_1(z)\|_{\mathcal{L}(H_\hbar^{-N},H_\hbar^{-N+1})} = O(\hbar|\mathrm{Im}(z)|^{-K_1}).$$

If we replace $Q_0(z)$ by

$$Q_L(z) = Q_0 + Q_0R_1 + \cdots + Q_0R_1^L \in \Psi^{-m}(M),$$

and denote $R_L = (R_1)^L \in \hbar^L\Psi^{-L}(M)$ we will get

$$(z-P)Q_L = (\mathrm{Id} - R_1)(\mathrm{Id} + R_1 + \cdots + R_1^L) = \mathrm{Id} - R_{L+1}(z)$$

with

$$\|R_L(z)\|_{\mathcal{L}(H_\hbar^{-N},H_\hbar^{-N+L})} = O(\hbar^L|\mathrm{Im}(z)|^{-K_L'})$$

for some $K_L' > 0$. and similarly, since the estimates for $Q_L(z)$ blow up as $\mathrm{Im}(z) \to 0$ only polynomially,

$$\|M_{\chi_1}Q_LM_{\chi_2}\|_{\mathcal{L}(H_\hbar^{-N},H_\hbar^N)} = O(\hbar^N|\mathrm{Im}(z)|^{-K_L''})$$

for some $K_L'' > 0$. As a consequence,

$$(z-P)^{-1} = Q_L + (z-P)^{-1}R_{L+1}(z),$$

and if we take L large enough, we will get

$$\|M_{\chi_1}(z-P)^{-1}M_{\chi_2}\|_{\mathcal{L}(H_\hbar^{-N},H_\hbar^N)} = O(\hbar^N|\mathrm{Im}(z)|^{-K_N})$$

as desired.

Step 2. We then prove (a), namely for any coordinate patch $(\varphi_\alpha, U_\alpha, V_\alpha)$, there exists $\chi \in C_0^\infty(U_\alpha)$ such that $(\varphi_\alpha^{-1})^* M_\chi f(P) M_\chi (\varphi_\alpha)^* \in \Psi^0(\mathbb{R}^n)$.

According to Beals's Theorem (for manifolds), it is enough to prove

$$\|\mathrm{ad}_{\widehat{l_1}^W} \circ \cdots \circ \mathrm{ad}_{\widehat{l_N}^W} (\varphi_\alpha^{-1})^* M_\chi f(P) M_\chi (\varphi_\alpha)^*\|_{\mathcal{L}(L^2)} = O(\hbar^N) \tag{10}$$

holds for any N and any linear functions $l_1, \cdots, l_N$ on $\mathbb{R}^{2n}$.

By definition one can check

$$\mathrm{ad}_{\widehat{l}^W}[(\varphi_\alpha^{-1})^* M_\chi (z-P)^{-1} M_\chi (\varphi_\alpha)^*] = (\varphi_\alpha^{-1})^* \mathrm{ad}_{\widehat{L}}[M_\chi (z-P)^{-1} M_\chi](\varphi_\alpha)^*$$

where

$$\widehat{L} = (\varphi_\alpha)^* M_{\widetilde{\chi}} \widehat{l}^W M_{\widetilde{\chi}} (\varphi_\alpha^{-1})^*$$

for some $\widetilde{\chi} \in C_0^\infty(V_\alpha)$ such that $\varphi_\alpha^* \widetilde{\chi} = 1$ on $\mathrm{supp}(\chi)$. Since

$$\|\mathrm{ad}_{\widehat{L}}[(z-P)^{-1}]\|_{\mathcal{L}(L^2)} = \|(z-P)^{-1} \mathrm{ad}_{\widehat{L}}(P)(z-P)^{-1}\|_{\mathcal{L}(L^2)} = O(\hbar |\mathrm{Im(z)}|^2),$$

we conclude from the Hellfer-Sjöstrand formula that

$$\|\mathrm{ad}_{\widehat{l}^W}[(\varphi_\alpha^{-1})^* M_\chi f(P) M_\chi (\varphi_\alpha)^*]\|_{\mathcal{L}(L^2)} = O(\hbar).$$

Similarly one can prove (10) for all N and the conclusion follows.

Step 3. Now we prove $f(P) \in \Psi^{-\infty}(M)$.

By Step 1 and Step 2, we have seen $f(P) \in \Psi^0(M)$. If we replace f by $g_k(t) = (t+i)^k f(t) \in \mathscr{S}$, we will get

$$g_k(P) = (P+i)^k f(P) : L^2(M) \to H_\hbar^k(M)$$

for any k and thus $f(P) \in \Psi^{-\infty}(M)$.

Step 4. Finally we calculate the principal symbol of $f(P)$.

Again this is a consequence of Hellfer-Sjöstrand formula. In Step 1 we have seen

$$\mathrm{Op}((z-p)^{-1})(z-P) = I + O(\hbar |\mathrm{Im}(z)|^{-K_1})$$

which implies

$$(z-P)^{-1} = \mathrm{Op}((z-p)^{-1}) + O(\hbar |\mathrm{Im}(z)|^{-K_1-1}).$$

Thus

$$f(P) = -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}_z \widetilde{f}(z) \mathrm{Op}((z-p)^{-1}) \mathcal{L}(dz) + O(\hbar) = \mathrm{Op}(f(p(x,\xi))) + O(\hbar).$$

So by definition, $\sigma(f(P)) = f(p(x,\xi))$. □

Note that in this setting the trace formula is even simpler: Under the ellipticity assumption on $p = \sigma(P)$ in $S^m(T^*M)$ with $m > 0$, locally the "principal symbol" of $f(P)$ is a Schwartz function. As a consequence, we get the following trace formula for $P \in \Psi^m(M)$ satisfying the conditions at the beginning of this section:

Theorem 3.3. *Suppose $f \in \mathscr{S}$. Then $f(P)$ is a trace class operator on $L^2(M)$ and*

$$(2\pi\hbar)^n \mathrm{Tr}(f(P)) = \int_{T^*M} f(p(x,\xi))dxd\xi + O(\hbar).$$

¶Generalized Weyl's law.

By playing the same trick, namely by approximating $\chi_{(a,b)}$ both from above and below via smooth functions, one can easily prove the following Weyl's law for $P \in \Psi^m(M)$ satisfying the conditions at the beginning of this section:

Theorem 3.4 (Weyl's law on manifolds)**.** *For any $a < b$, as $\hbar \to 0$ we have*

$$N_\hbar(P,[a,b]) = \frac{1}{(2\pi\hbar)^n}\Big(\mathrm{Vol}(p^{-1}([a,b])) + o(1)\Big).$$

In particular, if we take $P = -\hbar^2\Delta + V$, we will get Weyl's law for Schrödinger operator on manifolds,

$$N_\hbar(P,[a,b]) = \frac{1}{(2\pi\hbar)^n}\left(\mathrm{Vol}\{(x,\xi) \mid a \le |\xi|^2 + V(x) \le b\} + o(1)\right).$$

Remark. In particular, if we take $V = 0$, we get Weyl's law for the Laplace-Beltrami operator Δ on (M,g):

$$\#\{j \mid \lambda_j \le \lambda\} = \frac{\omega_n \mathrm{Vol}(M)}{(2\pi)^n}\lambda^{n/2} + o(\lambda^{n/2})$$

as $\lambda \to \infty$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

In fact, with a bit more work, we can prove the following generalized Weyl's law (which reduced to Weyl's law above if we take $B = Id$):

Theorem 3.5 (Generalized Weyl's law)**.** *Suppose $B \in \Psi^0(M)$ and $a < b$. Under the assumptions at beginning of this section, as $\hbar \to 0$ we have*

$$(2\pi\hbar)^n \sum_{a \le \lambda_j \le b} \langle B\varphi_j, \varphi_j\rangle \to \iint_{a \le p \le b} \sigma(B)dxd\xi. \tag{11}$$

Proof. Fix a and b and let $\Pi : L^2(M) \to L^2(M)$ be the orthogonal projection from $L^2(M)$ onto the subspace spanned by eigenfunction φ_j's of P associated to eigenvalues in the interval $[a,b]$. Then by definition,

$$\sum_{a \le \lambda_j \le b} \langle B\varphi_j, \varphi_j\rangle = \mathrm{Tr}(\Pi B \Pi).$$

As before we pick two sequence of compactly supported smooth functions $\underline{f_\varepsilon}$ and $\bar{f}_\varepsilon$ that approaches the characteristic function $\chi_{[a,b]}$ from below and above, namely

$$1_{[a+\varepsilon,b-\varepsilon]} \le \underline{f_\varepsilon} \le 1_{[a,b]} \le \bar{f}_\varepsilon \le 1_{[a-\varepsilon,b+\varepsilon]}.$$

Then $\underline{f_\varepsilon}(P)$ and $\bar{f}_\varepsilon(P)$ are pseudo-differential operators in $\Psi^{-\infty}(M)$ with

$$\Pi\underline{f_\varepsilon}(P) = \underline{f_\varepsilon}(P)\Pi = \underline{f_\varepsilon}(P), \quad \bar{f}_\varepsilon(P)\Pi = \Pi\bar{f}_\varepsilon(P) = \Pi,$$

and thus

$$\Pi B = \underline{f_\varepsilon}(P)B + (\Pi - \underline{f_\varepsilon}(P))B = \underline{f_\varepsilon}(P)B + \Pi(\bar{f}_\varepsilon(P) - \underline{f_\varepsilon}(P))B.$$

It follows

$$\mathrm{Tr}(\Pi B\Pi) = \mathrm{Tr}(\Pi B) = \mathrm{Tr}(\underline{f_\varepsilon}(P)B) + \mathrm{Tr}(\Pi(\bar{f}_\varepsilon(P) - \underline{f_\varepsilon}(P))B).$$

Since $0 \le \bar{f}_\varepsilon - \underline{f_\varepsilon} \le 1$ and since $\bar{f}_\varepsilon(P) - \underline{f_\varepsilon}(P)$ is self-adjoint, the definition of the trace norm together with Theorem 3.4 implies

$$\|\bar{f}_\varepsilon(P) - \underline{f_\varepsilon}(P)\|_{tr} \le \#\{j \mid \lambda_j \in (a-\varepsilon, a+\varepsilon) \cup (b-\varepsilon, b+\varepsilon)\} \le C(A_\varepsilon + B_\varepsilon(\hbar))\hbar^{-n},$$

where $A_\varepsilon = \mathrm{Vol}[p^{-1}((a-\varepsilon, a+\varepsilon) \cup (b-\varepsilon, b+\varepsilon))]$ and $\lim_{\hbar\to 0} B_\varepsilon(\hbar) = 0$. It follows [3]

$$(2\pi\hbar)^n\mathrm{Tr}\Big(\Pi(\bar{f}_\varepsilon(P) - \underline{f_\varepsilon}(P))B\Big) \le (2\pi\hbar)^n\|\Pi(\bar{f}_\varepsilon(P) - \underline{f_\varepsilon}(P))B\|_{tr} \le C(A_\varepsilon + B_\varepsilon(\hbar))$$

and thus

$$\begin{aligned}(2\pi\hbar)^n\mathrm{Tr}(\Pi B\Pi) &= (2\pi\hbar)^n\mathrm{Tr}(\underline{f_\varepsilon}(P)B) + C(A_\varepsilon + B_\varepsilon(\hbar)) \\ &= \int_{T^*M} f_\varepsilon(p)\sigma_0(B)dxd\xi + O_\varepsilon(\hbar) + C(A_\varepsilon + B_\varepsilon(\hbar)).\end{aligned}$$

It follows

$$\liminf_{\hbar\to 0}(2\pi\hbar)^n\mathrm{Tr}(\Pi B\Pi) = \int_{T^*M} f_\varepsilon(p)\sigma_0(B)dxd\xi + O(A_\varepsilon)$$

and

$$\limsup_{\hbar\to 0}(2\pi\hbar)^n\mathrm{Tr}(\Pi B\Pi) = \int_{T^*M} f_\varepsilon(p)\sigma_0(B)dxd\xi + O(A_\varepsilon).$$

Letting $\varepsilon \to 0$, the conclusion follows.

□

[3] Here we used the following fact: If A is compact and B is bounded, then their singular values satisfy $s_j(AB) \le s_j(A)\|B\|_{\mathcal{L}}$. This fact can be proved by using the min-max characterization of the eigenvalues of positive (self-adjoint) compact operators.