LECTURE 26: FIO – SYMPLECTIC GEOMETRY BACKGROUND

1. Symplectic structure on cotangent bundle

¶ Linear symplectic structure.

Definition 1.1. A *symplectic vector space* is a pair (V, Ω), where V is a real vector space, and $\Omega : V \times V \to \mathbb{R}$ a non-degenerate linear 2-form.[1] Ω is called a *linear symplectic structure* or a *linear symplectic form* on V.

Example. Let $V = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ and define

$$\Omega_0((x,\xi),(y,\eta)) := \langle x, \eta\rangle - \langle \xi, y\rangle,$$

then (V, Ω_0) is a symplectic vector space. Let $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$ be the standard basis of $\mathbb{R}^n \times \mathbb{R}^n$, and $\{e_1^*, \cdots, e_n^*, f_1^*, \cdots, f_n^*\}$ the dual basis of $(\mathbb{R}^n)^* \times (\mathbb{R}^n)^*$, then as a linear 2-form one has

$$\Omega_0 = \sum_{i=1}^n e_i^* \wedge f_i^*.$$

¶ Linear Darboux theorem.

Definition 1.2. Let (V_1, Ω_1) and (V_2, Ω_2) be symplectic vector spaces. A linear map $F : V_1 \to V_2$ is called a *linear symplectomorphism* if it is a linear isomorphism and

$$F^*\Omega_2 = \Omega_1. \tag{1}$$

Example. In Lecture 7 we have mentioned three simple linear symplectomorphisms $f : (\mathbb{R}^{2n}, \Omega_0) \to (\mathbb{R}^{2n}, \Omega_0)$:

- $f(x,\xi) = (-\xi, x)$.
- $f(x,\xi) = (x, \xi + Cx)$, where C is a symmetric $n \times n$ matrix.
- $f(x,\xi) = (Ax, (A^T)^{-1}x)$, where A is an invertible $n \times n$ matrix.

In fact, one can prove that any linear symplectomorphism is a composition of these simple ones.

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[1] Recall that a linear 2-form is a anti-symmetric bilinear map, namely $\Omega(u,v) = -\Omega(v,u)$. It it non-degenerate if

$$\Omega(u,v) = 0, \forall v \in \Omega \Longrightarrow u = 0.$$

Equivalently, the induced map

$$\widetilde{\Omega} : V \to V^*, \quad \widetilde{\Omega}(u)(v) = \Omega(u,v)$$

is bijective.

Of course linear symplectomorphism defines an equivalent relation between symplectic vector spaces. It turns out that up to linear symplectomorphism, $(\mathbb{R}^{2n}, \Omega_0)$ is the only $2n$-dimensional symplectic vector space:

Theorem 1.3 (Linear Darboux theorem)**.** *For any linear symplectic vector space* (V, Ω)*, there exists a linear symplectomorphism*

$$F : (V, \Omega) \to (\mathbb{R}^{2n}, \Omega_0).$$

Equivalently : given any symplectic vector space (V, Ω)*, there exists a dual basis* $\{e_1^*, \cdots, e_n^*, f_1^*, \cdots, f_n^*\}$ *of* V^* *so that as a linear 2-form,*

$$\Omega = \sum_{i=1}^{n} e_i^* \wedge f_i^*. \tag{2}$$

The basis is called a Darboux basis *of* (V, Ω).

Proof. Apply the Gram-Schmidt process with respect to the linear 2-form Ω. (For details, c.f. A. Canas de Silver, *Lectures on Symplectic Geometry*, page 1.) □

Remark. As a consequence, any symplectic vector space is even-dimensional.

Since a linear symplectic form is a linear 2-form, a natural question is: which 2-form in $\Lambda^2(V^*)$ is a linear symplectic form on V?

Proposition 1.4. *Let* V *be a* $2n$ *dimensional vector space. A linear 2-form* $\Omega \in \Lambda^2(V^*)$ *is a linear symplectic form on* V *if and only if as a* $2n$*-form,*

$$\Omega^n = \Omega \wedge \cdots \wedge \Omega \neq 0 \in \Lambda^{2n}(V^*). \tag{3}$$

Proof. If Ω is symplectic, then according to the linear Darboux theorem, one can choose a dual basis of V^* so that Ω is given by (2). It follows

$$\Omega^n = n! e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^* \neq 0.$$

Conversely, if Ω is degenerate, then there exists $u \in V$ so that $\Omega(u, v) = 0$ for all $v \in V$. Extend u into a basis $\{u_1, \cdots, u_{2n}\}$ of V with $u_1 = u$. Then since $\dim \Lambda^{2n}(V) = 1$, $u_1 \wedge \cdots \wedge u_{2n}$ is a basis of $\Lambda^{2n}(V)$. But $\Omega^n(u_1 \wedge \cdots \wedge u_{2n}) = 0$. So $\Omega^n = 0$. □

¶ Symplectic Manifolds: Definitions and examples.

Definition 1.5. A symplectic manifold is a pair (M, ω), where M is a smooth manifold, and $\omega \in \Omega^2(M)$ is a smooth 2-form on M, such that

(1) for each $p \in M$, $\omega_p \in \Lambda^2(T_p^* M)$ is a linear symplectic form on T_pM.
(2) ω is a closed 2-form, i.e. $d\omega = 0$

We call ω a *symplectic form* on M.

Remark. Note that if (M, ω) is symplectic, then $\dim M = \dim T_pM$ must be even. Denote $\dim M = 2n$. Then as we have seen,

$$\omega_p^n \neq 0 \in \Lambda^{2n}(T_p^*M),$$

i.e. ω^n is a non-vanishing $2n$ form, thus a *volume form* on M. We will call $\frac{\omega^n}{n!}$ the *symplectic volume form* or the *Liouville form* on M. As a simple consequence of the existence of a volume form, we see M must be orientable. (There are many other topological restriction for the existence of a symplectic structure. For example, S^{2n} ($n \geq 1$) does not admits any symplectic structure. In general it is very non-trivial to determine whether a manifold admits a symplectic structure.)

Still, we have plenty of interesting symplectic manifolds.

Example. $(\mathbb{R}^{2n}, \Omega_0)$ is of course the simplest symplectic manifold.

Example. Let S be any oriented surface and ω any volume form on S, then obviously (S, ω) is symplectic.

Example. Let X be any smooth manifold and $M = T^*X$ its cotangent bundle. We will see below that there exists a canonical symplectic form ω_{can} on M. So, we have "as many" symplectic manifolds as smooth manifolds!

¶ The canonical symplectic structure on cotangent bundles.

Let X be an n-dimensional smooth manifold and $M = T^*X$ its cotangent bundle. Let

$$\pi : T^*X \to X, \quad (x, \xi) \mapsto x$$

be the bundle projection map. From any coordinate patch $(\mathcal{U}, x_1, \cdots, x_n)$ of X one can construct a system of coordinates $(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$ on $M_{\mathcal{U}} = \pi^{-1}(\mathcal{U})$. Namely, if $\xi \in T_x^*X$, then

$$\xi = \sum \xi_i (dx_i)_x.$$

Using the computations at the beginning of Lecture 24, one can easily see that

$$\omega := \sum_{i=1}^n dx_i \wedge d\xi_i \tag{4}$$

is well-defined and is a symplectic form on $M = T^*X$.

Here is a coordinate free way to define ω: For any $p = (x, \xi) \in M$, we let

$$\alpha_p = (d\pi_p)^T \xi. \tag{5}$$

Note that by definition $\xi \in T_x^*X$, so for any $p \in T^*X$,

$$\alpha_p = (d\pi_p)^T \xi \in T_p^*(T^*X).$$

In other words, we get a (globally defined) smooth 1-form

$$\alpha \in \Omega^1(M) = \Gamma^\infty(T^*(T^*X)).$$

Definition 1.6. We call α the *canonical 1-form* (or *tautological 1-form*) on T^*X.

Proposition 1.7. *In local coordinates described above,*

$$\alpha = \sum_{i=1}^{n} \xi_i dx_i. \tag{6}$$

Proof. Let $v_p = \sum_{i=1}^n (a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial \xi_i}) \in T_pM$. Then

$$\langle \alpha_p, v_p \rangle = \langle \xi, (d\pi_p)v_p \rangle = \langle \sum \xi_i (dx_i)_x, \sum a_i \frac{\partial}{\partial x_i} \rangle = \sum a_i \xi_i = \langle \sum \xi_i dx_i, v_p \rangle.$$

The equation follows. □

As a consequence, if we let

$$\omega = -d\alpha, \tag{7}$$

then ω is closed, and is a symplectic form on M locally given by (4).

$$\omega = \sum dx_i \wedge d\xi_i$$

Definition 1.8. We call $\omega = -d\alpha$ the *canonical symplectic form* on $M = T^*X$.

A crucial property for the canonical 1-form $\alpha \in \Omega^1(M)$ is the following

Theorem 1.9 (Reproducing property)**.** *For any 1-form $\mu \in \Omega^1(X)$, if we let $s_\mu : X \to T^*X$ be the map that sends $x \in X$ to $\mu_x \in T_xX$, then we have*

$$s_\mu^* \alpha = \mu. \tag{8}$$

Conversely, if $\alpha \in \Omega^1(M)$ is a 1-form such that (8) hold for all 1-form $\mu \in \Omega^1(X)$, then α is the canonical 1-form.

Proof. At any point $p = (x, \xi)$ we have $\alpha_p = (d\pi_p)^T \xi$. So at $p = s_\mu(x) = (x, \mu_x)$ we have $\alpha_p = (d\pi_p)^T \mu_x$. It follows

$$(s_\mu^* \alpha)_x = (ds_\mu)_x^T \alpha_p = (ds_\mu)_x^T (d\pi_p)^T \mu_x = (d(\pi \circ s_\mu))_x^T \mu_x = \mu_x.$$

Conversely, suppose $\alpha_0 \in \Omega^1(M)$ is another 1-form on M satisfying the reproducing property above, then for any 1-form $\mu \in \Omega^1(X)$, we have $s_\mu^*(\alpha - \alpha_0) = 0$. So for any $v \in T_xX$,

$$0 = \langle (ds_\mu)_x^T (\alpha - \alpha_0)_p, v \rangle = \langle (\alpha - \alpha_0)_p, (ds_\mu)_x(v) \rangle.$$

For each $p = (x, \xi)$, the set of all vectors of the this form,

$$\{(ds_\mu)_x v \mid \mu \in \Omega^1(X), \mu_x = \xi, v \in T_xX, \}$$

span T_pM, so we conclude that $\alpha = \alpha_0$. □

¶ Symplectomorphisms.

As in the linear case, we can define

Definition 1.10. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. A smooth map $f : M_1 \to M_2$ is called a *symplectomorphism* (or a *canonical transformation*) if it is a diffeomorphism and

$$f^*\omega_2 = \omega_1. \tag{9}$$

We have the following amazing theorem for symplectic manifolds, whose proof can be found in A. Canas de Silver's book mentioned above:

Theorem 1.11 (Darboux theorem)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$. Then for any $p \in M$, there exists a coordinate patch $(\mathcal{U}, x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$ centered at p such that on $\mathcal{U}$,*

$$\omega = \sum dx_i \wedge d\xi_i.$$

The coordinate patch above is called a Darboux coordinate patch.

Remark. Equivalently, this says that one can find a neighborhood $\mathcal{U}$ near any point $p \in M$ so that $(\mathcal{U}, \omega)$ is symplectomorphic to (U, Ω_0), where U is some open neighborhood of 0 in $\mathbb{R}^{2n}$. So unlike Riemannian geometry, for symplectic manifolds there is no *local geometry*: locally all symplectic manifolds of the same dimension look the same. (However, there are much to say about the global geometry/topology of symplectic manifolds!)

Remark. For cotangent bundle $M = T^*X$ with the canonical symplectic form, we have seen that any coordinate patch on X gives a Darboux coordinate patch on M.

¶ Naturality.

The construction of the canonical symplectic form on cotangent bundles is natural in the following sense: Suppose X and Y are smooth manifolds of dimension n and $f : X \to Y$ a diffeomorphism. According to our computations at the beginning of Lecture 18, we can "lift" f to a map $\tilde{f} : T^*X \to T^*Y$ by

$$\tilde{f}(x, \xi) = (f(x), (df_x^T)^{-1}(\xi)). \tag{10}$$

Theorem 1.12 (Naturality)**.** *The map $\tilde{f} : T^*X \to T^*Y$ constructed above is a symplectomorphism with respect to the canonical symplectic forms.*

Proof. It is not hard to check that $\tilde{f}$ is a diffeomorphism. Denote the projections by $\pi_1 : T^*X \to X$ and $\pi_2 : T^*Y \to Y$. By definition

$$\pi_2 \circ \tilde{f} = f \circ \pi_1.$$

So if we denote $f(x, \xi) = (y, \eta)$, then

$$\tilde{f}^*\alpha_{T^*Y} = d\tilde{f}^T \circ (d\pi_2^T)\eta = (d\pi_1^T \circ df^T)\eta = (d\pi_1^T)\xi = \alpha_{T^*X}.$$

This of course implies $\tilde{f}^*\omega_{T^*Y} = \omega_{T^*X}$. □

¶ Hamiltonian theory.

Now suppose (M,ω) is a symplectic manifold and $H \in C^\infty(M)$ a smooth function. Then $dH \in \Omega^1(M)$ is a smooth 1-form on M. According to the non-degeneracy of ω, one can find a smooth vector field Ξ_H, called the *Hamiltonian vector field* on M so that

$$\iota_{\Xi_H}\omega = dH. \tag{11}$$

This gives an intrinsic definition of Ξ_H that we defined for T^*X via locally coordinates in Lecture 24.

The following properties are immediate.

Proposition 1.13. *For any smooth function $H \in C^\infty(M)$,*

1. $\mathcal{L}_{\Xi_H} H = 0$,
2. $\mathcal{L}_{\Xi_H}\omega = 0$.

Proof. We use the Cartan's magic formula $\mathcal{L}_X = d\iota_X + \iota_X d$:

(1) $\mathcal{L}_{\Xi_H} H = \iota_{\Xi_H} dH = \iota_{\Xi_H}\iota_{\Xi_H}\omega = \omega(\Xi_H, \Xi_H) = 0$.

(2) $\mathcal{L}_{\Xi_H}\omega = d\iota_{\Xi_H}\omega = ddH = 0$. □

Let $\rho_t : M \to M$ be the *Hamiltonian flow* generated by Ξ_H. Then we get

$$\frac{d}{dt}\rho_t^* H = \rho_t^* \mathcal{L}_{\Xi_H} H = 0,$$

i.e. $\rho_t^* H = H$. So the Hamiltonian H is invariant under the Hamilton flow of H. This is just the conservation law in the abstract symplectic setting. We can do similar computation use the second one,

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^* \mathcal{L}_{\Xi_H}\omega = 0,$$

i.e. $\rho_t^*\omega = \omega$. In other words, we proved

Theorem 1.14. *For any t, $\rho_t : M \to M$ is a symplectomorphism.*

As a consequence, as we mentioned in Lecture 24, the Liouville volume form is invariant under the Hamiltonian flow ρ_t.

2. Lagrangian submanifolds

¶ Linear subspaces in symplectic vector space.

Now we study interesting linear subspaces of a symplectic vector space (V, Ω).

Definition 2.1. The *symplectic ortho-complement* of a vector subspace $W \subset V$ is

$$W^\Omega = \{v \in V \mid \Omega(v,w) = 0 \text{ for all } w \in W\}. \tag{12}$$

Example. If $(V,\Omega) = (\mathbb{R}^{2n}, \Omega_0)$ and $W = \mathrm{span}\{e_1, e_2, f_1, f_3\}$, then

$$W^\Omega = \mathrm{span}\{e_2, f_3, e_4, \cdots, e_n, f_4, \cdots, f_n\}.$$

One can easily observe the difference the symplectic ortho-complement and the standard ortho-complement $W^\perp$ with respect to an inner product on V. For example, one always have $W \cap W^\perp = \{0\}$ while in most cases $W \cap W^\Omega \neq \{0\}$. However, W^Ω and $W^\perp$ do have the same dimensions:

Proposition 2.2. $\dim W^\Omega = 2n - \dim W$.

Proof. Let $\widetilde{W} = \mathrm{Im}(\widetilde{\Omega}|_W) \subset V^*$. Then $\dim \widetilde{W} = \dim W$. But we also have

$$W^\Omega = (\widetilde{W})^0 = \{u \in V : l(u) = 0 \text{ for all } l \in \widetilde{W}\}.$$

So the conclusion follows. □

Definition 2.3. A vector subspace W of a symplectic vector space (V, Ω) is called

- *isotropic* if $W \subset W^\Omega$.
  - Equivalently: $\Omega|_{W\times W} = 0$.
  - Equivalently: $\iota^*\Omega = 0 \in \Lambda^2(W^*)$, where $\iota : W \hookrightarrow V$ is the inclusion.
  - In particular $\dim W \leq \dim V/2$.
- *coisotropic* if $W \supset W^\Omega$.
  - Equivalently: W^Ω is isotropic.
  - In particular $\dim W \geq \dim V/2$.
- *Lagrangian* if $W = W^\Omega$.
  - Equivalently: W is isotropic and $\dim W = \dim V/2$.
  - Equivalently: W is coisotropic and $\dim W = \dim V/2$.
  - Equivalently: W is both isotropic and coisotropic.
  - In particular $\dim W = \dim V/2$.
- *symplectic* if $W \cap W^\Omega = \emptyset$.
  - Equivalently: $\Omega|_{W\times W}$ is a linear symplectic form on W.
  - Equivalently: W^Ω is a symplectic subspace.
  - In particular $\dim W$ is even.

Example. If $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$ is a Darboux basis of (V, Ω), then for any $0 \leq k \leq n$, $W_k = \mathrm{span}\{e_1, \cdots, e_k, f_{k+1}, \cdots, f_n\}$ is a Lagrangian subspace of (V, Ω).

Example. Let $F : (V_1, \Omega_1) \to (V_2, \Omega_2)$ be any linear symplectomorphism. Note that $\Omega = \Omega_1 \oplus (-\Omega_2)$ is a symplectic structure on $V = V_1 \oplus V_2$. It is easy to check that the graph of F,

$$\Gamma = \{(v_1, F(v_1)) \mid v_1 \in V_1\},$$

is a Lagrangian subspace of (V, Ω).

¶ Lagrangian submanifolds.

Similarly for symplectic manifolds, we can define

Definition 2.4. Let (M, ω) be a $2n$-dimensional symplectic manifold and $Z \subset M$ a submanifold. We call Z

- *isotropic* if for all $p \in Z$, T_pZ is an isotropic subspace of (T_pM, ω_p).

- *coisotropic* if for all $p \in Z$, T_pZ is a coisotropic subspace of (T_pM, ω_p).
- *Lagrangian* if for all $p \in Z$, T_pZ is a Lagrangian subspace of (T_pM, ω_p).
- *symplectic* if for all $p \in Z$, T_pZ is a symplectic subspace of (T_pM, ω_p).

One of the most important objects for us are the Lagrangian submanifolds. Even for the case of $(\mathbb{R}^{2n}, \Omega_0)$, Lagrangian submanifolds (not just linear Lagrangian subspaces) are fundamental objects and plays a crucial role in microlocal analysis.

By definition, a Lagrangian submanfold Γ of (T^*X, ω_{can}) has dimension $\dim \Gamma = \dim X$ and satisfies $\iota^*\omega_{can} = 0$, where $\iota : \Gamma \hookrightarrow T^*X$ is the inclusion map. From the local expression of ω, it is easy to see the following are Lagrangian submanifolds:

Example. For each $x \in X$, the cotangent fiber

$$T_x^*X = \{(x, \xi) \in T^*X \mid \xi \in T_x^*X\}$$

is a (vertical) Lagrangian submanifold of T^*X.

Example. The zero section of T^*X,

$$X_0 = \{(x, \xi) \in T^*X \mid x \in X, \xi = 0 \in T_x^*X\}$$

is a (horizontal) Lagrangian submanifold of T^*X.

Example. For any submanifold S of X, the conormal bundle

$$N^*S = \{(x, \xi) \in T^*X \mid x \in S, \xi \in T_x^*X \text{ so that } \xi(v) = 0 \text{ for all } v \in T_xS\}$$

is a Lagrangian submanifold of T^*X. Note: the previous two examples are special cases of this example.

¶ Horizontal Lagrangian submanifolds.

Let $M = T^*X$ be the cotangent bundle of any smooth manifold X and ω the canonical symplectic form on M. We have seen that the *zero section* $X_0 = \{(x, 0)\}$ is a *horizontal* Lagrangian submanifold of M. A natural question is: what are other *horizontal* Lagrangian submanifolds? Of course, an n dimensional submanifold of $M = T^*X$ is *horizontal* means it is the graph of a section $s_\mu : X \to T^*X$. In other words, any horizontal submanifold has the form

$$\Lambda_\mu = \{(x, \mu_x) \mid x \in X\},$$

where $\mu \in \Omega^1(X)$ is a smooth 1-form. When will this be Lagrangian? Let $\iota : \Lambda_\mu \hookrightarrow T^*X$ be the inclusion map. Note that $\pi \circ \iota : \Lambda_\mu \to X$ is a diffeomorphism. Let $\gamma : X \to \Lambda_\mu$ be its inverse. Then by definition

$$s_\mu = \iota \circ \gamma.$$

So from the reproducing property,

$$\Lambda_\mu \text{ is Lagrangian } \Leftrightarrow \iota^* d\alpha = 0 \Leftrightarrow \gamma^*\iota^* d\alpha = 0 \Leftrightarrow d(s_\mu^*\alpha) = 0 \Leftrightarrow d\mu = 0.$$

In other words, we proved

Proposition 2.5. *A horizontal submanifold Λ_μ is Lagrangian if and only if $d\mu = 0$.*

Definition 2.6. If μ is exact, i.e. $\mu = d\varphi$ for some smooth function $\varphi \in C^\infty(X)$, then we call φ a *generating function* of the Lagrangian submanifold Λ_μ.

¶ Lagrangian submanifolds v.s. symplectomorphisms.

Next let's study the relation between Lagrangian submanifolds and symplectomorphisms. Let (M_1, ω_1) and (M_2, ω_2) be $2n$ dimensional symplectic manifolds. Then on the product $M = M_1 \times M_2$ one has two important symplectic forms:

- the *product symplectic form* $\omega = \omega_1 \oplus \omega_2$,
- the *twisted product form* $\widetilde{\omega} = \omega_1 \oplus (-\omega_2)$.

Let $f : M_1 \to M_2$ be a diffeomorphism, then its graph

$$\Gamma_f = \{(x, f(x) \mid x \in M_1\}$$

is a $2n$ dimensional submanifold of the $4n$ dimensional manifold M.

Theorem 2.7. *f is a symplectomorphism if and only if Λ_f is Lagrangian with respect to $\widetilde{\omega}$.*

Proof. Let $\iota : \Gamma_f \hookrightarrow M$ be the inclusion and $\gamma : M_1 \to \Gamma_f \subset M$ be the diffeomorphism onto Γ_f, then

$$\Gamma_f \text{ is Lagrangian} \quad \Longleftrightarrow \iota^*\widetilde{\omega} = 0 \Longleftrightarrow \gamma^*\iota^*\widetilde{\omega} = 0 \Longleftrightarrow \omega_1 - f^*\omega_2 = 0.$$

□

In particular, suppose $M_1 = T^*X_1$ and $M_2 = T^*X_2$ be cotangent bundles and $\omega_1 = -d\alpha_1, \omega_2 = -d\alpha_2$ the canonical symplectic forms. Then $M = M_1 \times M_2 = T^*X$, where $X = X_1 \times X_2$. Moreover, the canonical 1-form on $M = T^*X$ is $\alpha = \alpha_1 \oplus \alpha_2$, so the product symplectic form $\omega = \omega_1 \oplus \omega_2$ on $M = T^*X$ is the canonical symplectic form. Let

$$\sigma_2 : M_2 \to M_2, (x, \xi) \mapsto (x, -\xi).$$

Then $\sigma_2^*\alpha_2 = -\alpha_2$, and thus $\sigma_2^*\omega_2 = -\omega_2$. It follows from the previous theorem that

Proposition 2.8. *If $f : M_1 \to M_2$ is a diffeomorphism, then f is a symplectomorphism if and only if $\Gamma_{\sigma_2 \circ f}$ is a Lagrangian submanifold of (M, ω).*

¶ Generating functions for symplectomorphisms.

Note that proposition 2.8 is equivalent to

The graph of f is a Lagrangian $\Longleftrightarrow \sigma_2 \circ f$ is a symplectomorphism.

From this correspondence it is natural to define

Definition 2.9. If $\Gamma_f = \Lambda_{d\varphi}$ for some $\varphi \in C^\infty(X_1 \times X_2)$, we say φ a generating function for the symplectomorphism $\sigma_2 \circ f$.

Remark. Usually one only need to find generating functions *locally.*

Now suppose we have a Lagrangian submanifold $\Lambda_{d\varphi} \subset T^*(X_1 \times X_2)$ generated by function $\varphi \in C^\infty(X_1 \times X_2)$. When will it generate a symplectomorphism $\sigma_2 \circ f$? In other words, we want $\Lambda_{d\varphi}$ to be the graph of some diffeomorphism $f : M_1 \to M_2$. We denote $pr_i : M = M_1 \times M_2 \to M_i$ be the projection maps. We choose local coordinate patches $(\mathcal{U}_1, x_1, \cdots, x_n)$ and $(\mathcal{U}_2, y_1, \cdots, y_n)$ on X_1 and X_2 respectively. Then $\Lambda_{d\varphi}$ is described locally by the equations $\xi_i = \frac{\partial\varphi}{\partial x_i}, \eta_i = \frac{\partial\varphi}{\partial y_i}$. Therefore, given any point $(x, \xi) \in M_1$, a point (y, η) is the image of (x, ξ) under the map f if and only if $(x, y, \xi, -\eta)$ is a point on the graph $\Lambda_{d\varphi}$. So to find $(y, \eta) = f(x, \xi)$, we need to solve the equations

$$\left\{ \begin{array}{l} \xi_i = \frac{\partial\varphi}{\partial x_i}(x, y), \\ \eta_i = -\frac{\partial\varphi}{\partial y_i}(x, y). \end{array} \right. \tag{13}$$

According to the implicit function theorem, to solve the first equation $\xi_i = \frac{\partial\varphi}{\partial x_i}(x, y)$ for y locally, we need the condition

$$\det \left[\frac{\partial^2\varphi}{\partial x_i \partial y_j} \right] \neq 0. \tag{14}$$

Of course after solving y we may feed it into the second equation to get η.

Example. Let $X_1 = X_2 = \mathbb{R}^n$ and $B = (b_{ij})$ a non-singular $n \times n$ matrix. Then the function $\varphi(x, y) = \sum b_{ij} x_i y_j$ generates a linear symplectomorphism $T_B : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ which maps (x, ξ) to $(B^{-1}\xi, -B^T x)$.

In particular, if $B = I$, i.e. $\varphi(x, y) = \sum x_i y_i$, then T_B maps (x, ξ) to $(\xi, -x)$.

Example. Let $X_1 = X_2 = \mathbb{R}^n$ and $\varphi(x, y) = -\frac{|x-y|^2}{2}$. Then equation (13) becomes

$$\left\{ \begin{array}{l} \xi_i = \frac{\partial\varphi}{\partial x_i}(x, y) = y_i - x_i \\ \eta_i = -\frac{\partial\varphi}{\partial y_i}(x, y) = y_i - x_i \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} y_i = x_i + \xi_i, \\ \eta_i = \xi_i. \end{array} \right.$$

So the symplectomorphism generated by φ is $f(x, \xi) = (x + \xi, \xi)$.

More generally, if X is a Riemannian manifold and $\varphi(x, y) = -\frac{d(x,y)^2}{2}$, where $d(x, y)$ is the Riemannian distance from x to y, then the symplectomorphism generated by φ is the geodesic flow.

Unfortunately not all Lagrangian submanifolds admits a generating function as described above. We will extend the conception of generating function by introducing "auxiliary variables" so that every Lagrangian submanifold of T^*X is locally represented by a generating function.