LECTURE 27: FIO – SYMPLECTIC CATEGORY

1. Basics on Category

¶ Category and functor.

Definition 1.1 (Category)**.** A *category* $\mathcal{C}$ consists of

(1) a class $\mathrm{Ob}(\mathcal{C})$ whose elements are called *objects*,
(2) a class $\mathrm{Mor}(\mathcal{C})$ of *morphisms* between the objects, such that
   - Each morphism f has a *source object* $X \in \mathrm{Ob}(\mathcal{C})$ and a *target object* $Y \in \mathrm{Ob}(\mathcal{C})$. We write $f : X \to Y$ and say "f is a morphism from X to Y". We denote all morphisms from X to Y by $\mathrm{Mor}(X, Y)$. [1]
   - The composition of morphisms $f : X \to Y$ and $g : Y \to Z$ is a morphism $g \circ f : X \to Z$, such that
     (a) (associativity) if $f : X \to Y, g : Y \to Z$ and $h : Z \to W$, then
     $$h \circ (g \circ f) = (h \circ g) \circ f,$$
     (b) (identity) for any $X \in \mathrm{Ob}(\mathcal{C})$, there is an *identity morphism* $\mathrm{Id}_X : X \to X$, such that for any morphism $f : Z \to X$ and $g : X \to Y$,
     $$\mathrm{Id}_X \circ f = f \quad \text{and} \quad g \circ \mathrm{Id}_X = g.$$

Definition 1.2. A (covariant) *functor* $\mathfrak{F}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$

(1) associates to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$,
(2) associates to each morphism $f : X \to Y$ in $\mathcal{C}$ a morphism $\mathfrak{F}(f) : \mathfrak{F}(X) \to \mathfrak{F}(Y)$ in $\mathcal{D}$ such that the following two conditions hold:
   - $\mathfrak{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathfrak{F}(X)}$ for every object X in $\mathcal{C}$,
   - $\mathfrak{F}(g \circ f) = \mathfrak{F}(g) \circ \mathfrak{F}(f)$ for any morphism $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$.

Example. We have the category of topological spaces $\mathcal{TOP}$,

- $\mathrm{Ob}(\mathcal{TOP})$ = all topological spaces,
- morphisms are continuous maps between topological spaces,

and the category of groups $\mathcal{GROUP}$,

- $\mathrm{Ob}(\mathcal{GROUP})$ = all groups,
- morphisms are group homomorphisms.

Moreover, we have many functors from $\mathcal{TOP}$ to $\mathcal{GROUP}$: π_n, H^n, H_n etc which were studied extensively in algebraic topology.

[1] It is possible that for some objects X and Y, there is no morphism from X to Y, so that $\mathrm{Mor}(X, Y) = \emptyset$.

Example. One can define a category $\widetilde{\mathcal{M}}$ so that

- $\mathrm{Ob}(\widetilde{\mathcal{M}})$ = all smooth manifolds,
- $\mathrm{Mor}(X, Y)$ = diffeomorphisms from X to Y,

and a category $\widetilde{\mathcal{S}}$ so that

- $\mathrm{Ob}(\widetilde{\mathcal{S}})$ = all symplectic manifolds,
- $\mathrm{Mor}(M, N)$ = symplectomorphisms from M to N.

Then the map $\mathfrak{F}$ that sends X to (T^*X, ω_{T^*X}) and sends a diffeomorphism f to its lifting $\tilde{f}$ is a functor. Unfortunately the categories $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{S}}$ have an obvious shortage: for most pairs of objects, there is no morphism at all!

Here is one way to get a more reasonable category $\mathcal{M}$ of smooth manifolds:

- $\mathrm{Ob}(\mathcal{M})$ = all smooth manifolds,
- $\mathrm{Mor}(X, Y)$ = smooth maps from X to Y.

But, what is a reasonable way to define a category of symplectic manifolds?

¶ The category $\mathcal{SETS}$.

In many cases, it is natural to define a morphism from X to Y to be a map from X to Y that satisfies specific conditions, so that the composition of morphisms can be defined by composition, and the identity morphism is just the identity map. However, there does exist interesting examples where morphisms are not maps:

Example. Let's define a category $\mathcal{SETS}$ as follows:

- $\mathrm{Ob}(\mathcal{SETS})$ = "all" sets (with some restriction to avoid logical issues.)
- $\mathrm{Mor}(X, Y)$ = all subsets of $X \times Y$.

Recall that a subset of $X \times Y$ is also called a *relation*. So we are using relations instead of maps to define morphisms. And, we can define the composition of morphisms to be the composition of relations:

- For $\Gamma_1 \in \mathrm{Mor}(X, Y)$ and $\Gamma_2 \in \mathrm{Mor}(Y, Z)$, the composition is

$$\Gamma_2 \circ \Gamma_1 = \{(x, z) \in X \times Z \mid \exists y \in Y \text{ so that } (x, y) \in \Gamma_1, (y, z) \in \Gamma_2.\} \tag{1}$$

It is not hard to prove the associativity $(\Gamma_3 \circ \Gamma_2) \circ \Gamma_1 = \Gamma_3 \circ (\Gamma_2 \circ \Gamma_1)$, and it is also clear that the identity morphism is

$$\mathrm{Id}_X = \{(x, x) \in X \times X \mid x \in X\} = \Delta_X. \tag{2}$$

Moreover, any relation $\Gamma \subset X \times Y$ has a natural "transpose"

$$\Gamma^T = \{(y, x) \mid (x, y) \in \Gamma\} \subset Y \times X. \tag{3}$$

Finally if we define a category $\widetilde{\mathcal{SETS}}$ of sets by the usual way,

- $\mathrm{Ob}(\widetilde{\mathcal{SETS}})$ = "all" sets (with some restriction to avoid logical issues),

- $\mathrm{Mor}(X, Y)$ = maps from X to Y,

then one can embed $\widetilde{\mathcal{SETS}}$ into $\mathcal{SETS}$ by a functor $\mathfrak{F}$ that maps any set to itself and maps any map $f : X \to Y$ to its graph

$$\Gamma_f = \{(x, f(x)) \mid x \in X\}.$$

In other words, the category $\mathcal{SETS}$ *enlarges* the category $\widetilde{\mathcal{SETS}}$.

¶ Categorical "points".

In the category $\mathcal{SETS}$ (and in all examples above), there is a distinguished object: the one point set "*pt*". In a category with such a distinguished object "*pt*", one can define for every object X in this category the "points" of X:

Definition 1.3. A *categorical point* of X is a morphism $\Gamma : pt \to X$.

Note that every morphism $\Gamma : X \to Y$ induces a map Γ^* from "points" of X to "points" of Y by composition.

Example. In the categories $\widetilde{\mathcal{SETS}}$, $\mathcal{M}$, $\widetilde{\mathcal{M}}$, $\widetilde{\mathcal{S}}$, the categorical points are the points in the usual sense. However, in the category $\mathcal{SETS}$, a categorical point of a set X is a subset of X.

2. Linear Symplectic Category

¶ The category $\mathcal{LS}$.

We can define the category $\widetilde{\mathcal{LS}}$ to be

- $\mathrm{Ob}(\widetilde{\mathcal{LS}})$ = symplectic vector spaces ,
- $\mathrm{Mor}((V, \Omega_V), (W, \Omega_W))$ = linear symplectomorphisms from (V, Ω_V) to (W, Ω_W).

Obviously this is not very interesting, because the set of morphisms between two symplectic vector spaces is non-empty only if they have the same dimension. In what follows we will define a slightly *larger* category, $\mathcal{LS}$, in which the set of morphisms between any two symplectic vector spaces is non-empty. However we prefer not to use all *relations* as morphisms as in the example $\mathcal{SETS}$ above – we want our morphisms to be geometrically interesting.

For simplicity we will abbreviate (V, Ω_V) to V, and abbreviate $(V, -\Omega_V)$ to V^-. By this way, $V \times W^-$ means the symplectic vector space $(V \oplus W, \Omega_V \oplus (-\Omega_W))$. We have shown in lecture 26 that the graph of any linear symplectomorphism from V to W is a Lagrangian subspace of $V \times W^-$. On the other hand, $V \times W^-$ has more Lagrangian subspaces than those arise from the graphs of linear symplectomorphisms. Moreover, for the existence of Lagrangian subspace in $V \times W^-$ we don't require $\dim V = \dim W$. Note that a Lagrangian subspace of $V \times W^-$ is a special subset of $V \times W$, and thus a special *relation* from V to W.

Definition 2.1. A Lagrangian subspace Γ of $V \times W^-$ is called a *linear canonical relation* from V to W.

Using this language we can define the category $\mathcal{LS}$ as

- $\mathrm{Ob}(\mathcal{LS})$ = symplectic vector spaces,
- $\mathrm{Mor}(V, W)$ = linear canonical relations from V to W.

It remains to prove that $\mathcal{LS}$ *is* a category, i.e. the composition of two linear canonical relations is a linear canonical relation. (One also need to prove the associativity, but let's ignore this.) Before we prove this let's first admit that $\mathcal{LS}$ is a category and describe its categorical points: The distinguished object "*pt*" is just the 0-dimensional vector space $\{0\}$. So for any symplectic vector space V, its categorical points are the linear canonical relations $\Gamma \subset \{0\} \times V^-$. They are of course in one-to-one correspondence with the Lagrangian subspaces of V. In other words, in the category $\mathcal{LS}$,

$$\boxed{\text{Categorical points of } V = \text{Lagrangian subspaces of } V.}$$

A. Weinstein gave a physical interpretation of this: According to Heisenberg's uncertainty principle, if you specify the position of a quantum particle, i.e. if you insists that it lie on the Lagrangian submanifold, $x_i = a_i, 1 \le i \le n$, of $\mathbb{R}^{2n} = T^*\mathbb{R}^n$, you forfeit all hope of knowing about its momentums. Similar statement for specifying the momentum. In other words, in quantum mechanics, points of $\mathbb{R}^{2n}$ are irrelevant. Instead, the Lagrangians, i.e. the *categorical points*, are relevant.

¶ Composition of linear canonical relations.

Let $\Gamma_1 \subset U \times V^-$ and $\Gamma_2 \subset V \times W^-$ be two linear canonical relations. Their composition is defined in analogy to the composition of relations in $\mathcal{SETS}$, i.e.

$$\Gamma_2 \circ \Gamma_1 = \{(u, w) \in U \times W \mid \exists v \in V \text{ such that } (u, v) \in \Gamma_1, (v, w) \in \Gamma_2\} \tag{4}$$

Our goal is to prove

Theorem 2.2. *$\Gamma_2 \circ \Gamma_1 \subset U \times W^-$ is a linear canonical relation.*

Before we prove this, we will first prove a special case where $U = \{0\}$, i.e.

Theorem 2.3. *Suppose $\Gamma \subset V \times W^-$ is a linear canonical relation and $\Lambda \subset V$ a Lagrangian subspace. Then $\Gamma(\Lambda)$ is a Lagrangian subspace of W.*

In other words, linear canonical relations "maps" "points" to "points".

Proof. If we denote $\pi_1 : V \times W \to V$ and $\pi_2 : V \times W \to W$ be the projections, then

$$\Gamma(\Lambda) = \pi_2((\Lambda \times W) \cap \Gamma) = \pi_2(\pi_1^{-1}(\Lambda) \cap \Gamma).$$

So $\Gamma(\Lambda)$ is a vector subspace of W. It remains to prove

(1) $\Gamma(\Lambda)$ is isotropic in W.
(2) $\dim \Gamma(\Lambda) = \frac{1}{2} \dim W$.

The proof of (1): Suppose $w_1, w_2 \in \Gamma(\Lambda)$, then by definition, there exists $v_1, v_2 \in \Lambda$ so that $\gamma_i = (v_i, w_i) \in \Gamma, i = 1, 2$. For simplicity we will denote $\Omega = \Omega_V \oplus (-\Omega_W)$. Since $\Lambda \subset V$ and $\Gamma \subset V \times W^-$ are both Lagrangians,

$$\Omega_V(v_1, v_2) = 0, \quad \Omega(\gamma_1, \gamma_2) = 0.$$

On the other hand,

$$\Omega(\gamma_1, \gamma_2) = \Omega_V(v_1, v_2) - \Omega_W(w_1, w_2).$$

It follows $\Omega_W(w_1, w_2) = 0$.

The proof of (2): Let $H = \pi_1^{-1}(\Lambda) = \Lambda \times W \subset V \times W$, then $H^\Omega = \Lambda \times \{0\}$. Consider the map

$$\alpha = \pi_2 \circ \iota_{H\cap\Gamma} : H \cap \Gamma \hookrightarrow V \times W \to W,$$

then $\mathrm{Im}(\alpha) = \pi_2(H \cap \Gamma) = \Gamma(\Lambda)$, and

$$\ker(\alpha) = \{(v, 0) \in H \cap \Gamma \mid v \in \Lambda\} = H^\Omega \cap \Gamma.$$

Similarly if we let τ be the map

$$\tau : H \times \Gamma \to V \times W, \quad (h, \gamma) \mapsto h - \gamma,$$

then $\mathrm{Im}(\tau) = H + \Gamma$. It follows

$$(\mathrm{Im}(\tau))^\Omega = H^\Omega \cap \Gamma = \ker(\alpha).$$

In particular, $\dim\ker(\alpha) = \dim\mathrm{coker}(\tau)$, and thus

$$\dim\mathrm{Im}(\alpha) = \dim H \cap \Gamma - \dim\mathrm{coker}(\tau).$$

To calculate the right hand side, we consider the short exact sequence

$$0 \to H \cap \Gamma \overset{\Delta}{\hookrightarrow} H \times \Gamma \overset{\tau}{\to} V \times W \to \mathrm{coker}(\tau) \to 0.$$

The exactness (which is easy to check) implies

$$\dim H \cap \Gamma + \dim V \times W = \dim H \times \Gamma + \dim\mathrm{coker}(\tau).$$

It follows

$$\dim H \cap \Gamma - \dim\mathrm{coker}(\tau) = \dim H + \dim\Gamma - \dim V - \dim W = \frac{1}{2}\dim W.$$

□

Proof of Theorem 2.2. We let $\widetilde{V} = U \times V^-$ and $\widetilde{W} = U \times W^-$. Then we can identify

$$\widetilde{V} \times \widetilde{W}^- = (U \times U^-) \times (V \times W^-)^-.$$

Since $\Lambda := \Gamma_1$ is a Lagrangian subspace of $\widetilde{V}$, and

$$\Gamma := \Delta_U \times \Gamma_2 \subset \widetilde{V} \times \widetilde{W}^-$$

is a linear canonical relation, Theorem 2.3 implies that

$$\Gamma(\Lambda) = \{(u, w) \in U \times W \mid \exists (u, v) \in \Gamma_1 \text{ such that } (u, u, v, w) \in \Gamma\} = \Gamma_2 \circ \Gamma_1$$

is a Lagrangian subspace of $\widetilde{W}$. □

3. Symplectic "Category"

¶ The "category" $\mathcal{S}$.

Motivated by the category $\mathcal{LS}$ of symplectic vector spaces, one can define a category $\mathcal{S}$ of symplectic manifolds via

- $\mathrm{Ob}(\mathcal{S})$ = symplectic manifolds,
- $\mathrm{Mor}(M,N)$ = canonical relations from M to N,

where, as in the linear case, we use the notion

Definition 3.1. A *canonical relation* from a symplectic manifold M to a symplectic manifold N is a Lagrangian submanifold of $M \times N^-$, where $N^- = (N, -\omega_N)$.

BAD NEWS: Given canonical relations $\Gamma_1 \in M_1 \times M_2^-$ and $\Gamma_2 \in M_2 \times M_3^-$, their composition $\Gamma_2 \circ \Gamma_1$ may fail to be a smooth submanifold, and as a result, fail to be a canonical relation from M_1 to M_3! As a consequence, $\mathcal{S}$ is NOT a true category.

¶ Transversal/Clean intersection conditions.

Recall from manifold theory the following useful criteria for a submanifold:

Definition 3.2. Suppose X, Z are smooth manifolds and Y a submanifold of Z. Let $f : X \to Z$ be a smooth map. We say f intersects Y *transversally* if

$$\mathrm{Im}(df_p) + T_{f(p)}Y = T_{f(p)}Z, \qquad \forall p \in f^{-1}(Y). \tag{5}$$

Theorem 3.3. *If f intersects Y transversally, then $f^{-1}(Y)$ is a smooth submanifold of X whose tangent space at $p \in f^{-1}(Y)$ is*

$$T_p(f^{-1}(Y)) = (df_p)^{-1}(T_{f(p)}Y). \tag{6}$$

There are two special cases of this definition/theorem:

- Suppose X, Y smooth submanifolds of Z. We say X and Y intersect *transversally* in Z if $\iota : X \hookrightarrow Z$ intersect Y transversally. Equivalently:

$$T_pZ = T_pX + T_pY, \qquad \forall p \in X \cap Y. \tag{7}$$

  In this case, $X \cap Y$ is a smooth submanifold of Z whose tangent space is

$$T_p(X \cap Y) = T_pX \cap T_pY. \tag{8}$$

- Suppose $f_i : X_i \to Z$, $i = 1, 2$ are smooth maps. We say f_1 and f_2 intersect *transversally* if the product map $f_1 \times f_2 : X_1 \times X_2 \to Z \times Z$ intersects with the diagonal $\Delta_Z = \{(z, z) \mid z \in Z\}$ transversally. In this case, the *fiber product*

$$F = (f_1 \times f_2)^{-1}(\Delta_Z) = \{(x_1, x_2) \mid f_1(x_1) = f_2(x_2)\} \tag{9}$$

  is a smooth submanifold of $X_1 \times X_2$ whose tangent space at $x = (x_1, x_2) \in F$ is

$$T_xF = \{(v_1, v_2) \mid v_i \in T_{x_i}X_i, (df_1)_{x_1}(v_1) = (df_2)_{x_2}(v_2)\}. \tag{10}$$

Remark. Of course the most important thing for us is the conclusions in the above theorems, not the conditions. This motivates the following definition:

Definition 3.4. Let X, X_i, Y, Z be smooth manifolds, and f, f_i smooth maps.

(1) We say $f : X \to Z$ intersects $Y \subset Z$ *cleanly* if $f^{-1}(Y)$ is a smooth submanifold of Y and for all $p \in f^{-1}(Y)$, $T_p(f^{-1}(Y)) = (df_p)^{-1}(T_{f(p)}Y)$.
(2) We say $X, Y \subset Z$ intersect *cleanly* in Z if $\iota : X \hookrightarrow Z$ intersect Y cleanly.
(3) We say f_1 and f_2 intersect *cleanly* if the product map $f_1 \times f_2$ intersects with the diagonal $\Delta_Z \subset Z \times Z$ cleanly.

Note that for (2) and (3), the tangent space are given by (8) and (10) respectively.

¶ Composition of canonical relations under cleanness condition.

Now suppose M_i are symplectic and $\Gamma_i \subset M_i \times M_{i+1}^-$ are canonical relations. Consider the projections

$$\pi : \Gamma_1 \to M_2, \quad (m_1, m_2) \mapsto m_2$$
$$\rho : \Gamma_2 \to M_2, \quad (m_2, m_3) \mapsto m_2.$$

Theorem 3.5. *If π and ρ intersect cleanly, then $\Gamma_2 \circ \Gamma_1$ is an immersed canonical relations in $M_1 \times M_3^-$.*

Proof. Since π and ρ intersect cleanly, the fiber product

$$F = (\pi \times \rho)^{-1}(\Delta_{M_2}) \simeq \{(m_1, m_2, m_3) \mid (m_1, m_2) \in \Gamma_1, (m_2, m_3) \in \Gamma_2\}$$

is a submanifold of $M_1 \times M_2 \times M_3$ whose tangent space at $m = (m_1, m_2, m_3)$ equals

$$T_m F = \{(v_1, v_2, v_3) \mid v_i \in T_{m_i} M_i, (v_1, v_2) \in T_{(m_1,m_2)}\Gamma_1, (v_2, v_3) \in T_{(m_2,m_3)}\Gamma_2\}.$$

Let $\iota : F \hookrightarrow M_1 \times M_2 \times M_3$ be the inclusion, and

$$\kappa : M_1 \times M_2 \times M_3 \to M_1 \times M_3, \quad (m_1, m_2, m_3) \mapsto (m_1, m_3)$$

be the "projection onto the first and third component" map. Then by definition,

$$\Gamma_2 \circ \Gamma_1 = \mathrm{Im}(\kappa \circ \iota).$$

Note that $d(\kappa \circ \iota)_m$ is just the projection map

$$d(\kappa \circ \iota)_m : T_m F \to T_{(m_1,m_3)}(M_1 \times M_3), \quad (v_1, v_2, v_3) \mapsto (v_1, v_3).$$

This has two implications:

- By definition,
$$T_{(m_1,m_3)}(\mathrm{Im}(\kappa \circ \iota)) = \mathrm{Im}(d(\kappa \circ \iota)_m) = T_{(m_2,m_3)}\Gamma_2 \circ T_{(m_1,m_2)}\Gamma_1$$
is a linear canonical relation in $T_{m_1} M_1 \times T_{m_3} M_3$.
- $\kappa \circ \iota$ is a constant rank map, so its mage $\Gamma_2 \circ \Gamma_1$ is an immersed submanifold of $M_1 \times M_3$.

It follows that $\Gamma_2 \circ \Gamma_1$ is an immersed Lagrangian submanifold in $M_1 \times M_3^-$. □

Fact from manifold theory: the image of the constant rank map $\kappa \circ \iota$ is an embedded submanifold if $\kappa \circ \iota$ is proper and its level sets are connected. (For a proof, see the appendix to this lecture.) So

Theorem 3.6. *Suppose π and ρ intersect cleanly. In addition suppose $\kappa \circ \iota$ is proper and for any $(m_1, m_3) \in \Gamma_2 \circ \Gamma_1$, $(\kappa \circ \iota)^{-1}(m_1, m_3)$ is connected. Then $\Gamma_2 \circ \Gamma_1$ is a canonical relation in $M_1 \times M_3^-$.*

¶ The category $\mathcal{M}$ v.s. the "category" $\mathcal{S}$.

Recall that the category $\mathcal{M}$ of smooth manifolds consists

- $\mathrm{Ob}(\mathcal{M})$ = all smooth manifolds,
- $\mathrm{Mor}(X, Y)$ = smooth maps from X to Y.

We also learned that any object X in $\mathcal{M}$ corresponds to an object $M = T^*X$ in $\mathcal{S}$. Now suppose $f : X \to Y$ is a morphism, i.e. a smooth map. Then its graph X_f is a submanifold of $X_1 \times X_2$, associated to which we have a Lagrangian submanifold N^*X_f of $T^*X_1 \times T^*X_2$. If we let σ be the involution

$$\sigma : T^*X_1 \times T^*X_2 \to T^*X_1 \times T^*X_2, \quad (x, \xi, y, \eta) \mapsto (x, \xi, y, -\eta),$$

then $\Gamma_f = \sigma(N^*X_f)$ is a canonical relation. So we get an *embedding* (a covariant "functor") of category $\mathcal{M}$ into "category" $\mathcal{S}$:

- $X \mapsto M = T^*X$.
- $(f : X \to Y) \mapsto \Gamma_f = \sigma(N^*X_f)$.

¶ Composition of a sympectomorphism with a canonical relation.

Suppose $\Gamma_1 \in M_1 \times M_2$ is a canonical relation and Γ_2 is the graph of a symplectomorphism from M_2 to M_3. Then

$$d\rho_{(m_2,m_3)} : T_{(m_2,m_3)}\Gamma_2 \to T_{m_2}M_2$$

is surjective. It follows that for each $m = (m_1, m_2, m_2, m_3) \in \Gamma_1 \times \Gamma_2$, the image of $d(\pi \times \rho)_m$ contains all vectors of the form $(0, w) \in T_{m_2}M_2 \times T_{m_2}M_2$. In other words, $\pi \times \rho$ intersects the diagonal Δ_{M_2} transversally. So the fiber product F is a smooth submanifold. Moreover, the map κ is given by

$$\kappa : (m_1, m_2, g(m_2)) \mapsto (m_1, g(m_2)),$$

where $g : M_2 \to M_3$ is the symplectomorphism. Since g is one-to-one, so is κ. It follows that $\Gamma_2 \circ \Gamma_1$ is a canonical relation. Similar argument holds if Γ_1 is defined by a symplectomorphism. In conclusion, we showed

Theorem 3.7. *The composition of the graph of a symplectomorphism with a canonical relation is again a canonical relation.*

In particular, we see that the composition of a canonical relation $\Gamma \in M \times N^-$ with the diagonal Δ_M is again a canonical relation.

4. Appendix

In this appendix we prove

Theorem 4.1. *Let $f : M \to N$ be a constant rank proper map and each level set is connected. Then the image of f is an embedded submanifold on N.*

Proof. We know that the image of a constant rank map is an immersed submanifold. In other words, for any $x \in M$, there exist open neighborhoods U_x of x in M and $V_{f(x)}$ of $f(x)$ in N such that $f(U_x)$ is an embedded submanifold of N. In what follows we try to find open neighborhood $\widetilde{V}_{f(x)} \subset V_{f(x)}$ of $f(x)$ in N such that

$$f(M) \cap \widetilde{V}_{f(x)} = f(U_x) \cap \widetilde{V}_{f(x)},$$

from which we conclude that $f(M)$ is an embedded submanifold of N.

We will proceed by contradiction. We shall need the following fact:

> **Fact**: If $f(x') = f(x)$, then for any open neighborhood U_x and $U_{x'}$, there exists open neighborhood $\widetilde{U}_x \subset U_x$ and $\widetilde{U}_{x'} \subset U_{x'}$ so that $f(\widetilde{U}_x) = f(\widetilde{U}_{x'})$.
>
> *Proof.* We define an equivalence relation on $f^{-1}(f(x))$ by
>
> $$x \sim x' \text{ if the property in the statement holds for } x \text{ and } x'.$$
>
> Then by local structure theorem of constant rank map, each equivalence class is open in $f^{-1}(f(x))$. Since the complement of any equivalence class is a union of equivalence classes, each equivalence class is also closed in $f^{-1}(f(x))$. Thus by the connectedness of $f^{-1}(f(x))$, there is only one equivalence class. □

Now suppose no such $\widetilde{V}_{f(x)}$ exists. Then there exists $y_k = f(x_k) \in f(X)$ but $y_k \notin f(U_x)$ such that $y_k \to f(x)$. Since f is proper and $\{y_k, f(x)\}$ is compact, x^k's lies in a compact set in M and thus have a convergent subsequence $x_{k_i} \to x' \in f^{-1}(f(x))$. According to the fact we just proved, there exists open neighborhood $\widetilde{U}_x \subset U_x$ and $\widetilde{U}_{x'}$ so that $f(\widetilde{U}_x) = f(\widetilde{U}_{x'})$. It follows that for k large enough,

$$y_k \in f(\widetilde{U}_{x'}) = f(\widetilde{U}_x) \subset f(U_x),$$

a contradiction. □