LECTURE 28: FIO – THE ENHANCED SYMPLECTIC CATEGORY

1. The calculus of densities

First let's recall the change of variables formula from calculus: if f is an integrable function defined in domain $U \subset \mathbb{R}^n$, $\varphi : U' \to U$ a bijective smooth map from U' to U, with $x = \varphi(x')$, then

$$\int_U f(x)dx = \int_{U'} f(\varphi(x')) \left| \frac{\partial x}{\partial x'} \right| dx',$$

where $\frac{\partial x}{\partial x'}$ is the Jacobian matrix of the coordinate change $x' \to x = \varphi(x')$. It is this factor $\left|\frac{\partial x}{\partial x'}\right|$ that motivates the conception of densities.

¶ Densities on vector space.

Let V be a vector space of dimension n. We denote by $\mathcal{F}(V)$ the set of all bases of V. Then for any two bases $\{e_i\}$ and $\{f_i\}$ of V, there exists a unique $A \in GL(n, \mathbb{R})$ that maps $\{e_i\}$ to $\{f_i\}$.

Definition 1.1. Let $\alpha \in \mathbb{C}$ be a complex number. An *α-density* on V is a map $\mu : \mathcal{F}(V) \to \mathbb{C}$ such that for any $v_i \in V$ and $A \in End(V)$,

$$\mu(Av_1, \cdots, Av_n) = |\det A|^\alpha \; \mu(v_1, \cdots, v_n) \tag{1}$$

We will denote the space of α-densities on V by $|V|^\alpha$.

Remark. An n-form on V is a map $\omega : V^n \to \mathbb{C}$ such that

$$\omega(Av_1, \cdots, Av_n) = (\det A)\; \omega(v_1, \cdots, v_n).$$

So if $\omega \in \Lambda^n(V)$ is an n-form, $|\omega|$ is a 1-density. ($\rightsquigarrow |\omega|^\alpha$ is an α-density.)

We list a couple properties of α-densities:

Proposition 1.2. *Let V, V', V'', W be vector spaces.*

1. *$|V|^\alpha$ is a one dimensional vector space over $\mathbb{C}$.*
2. *There is a canonical isomorphism $|V|^\alpha \otimes |V|^\beta \simeq |V|^{\alpha+\beta}$.*
3. *There is a canonical anti-linear isomorphism $|V|^\alpha \simeq |V|^{\bar\alpha}$.*
4. *Any short exact sequence $0 \to V' \to V \to V'' \to 0$ induces a canonical isomorphism $|V|^\alpha \simeq |V'|^\alpha \otimes |V''|^\alpha$. (Similar results holds for long exact sequence).*
5. *$|V|^\alpha \simeq |V^*|^{-\alpha}$.*

(6) *Any linear isomorphism $L : V \to W$ induces a pull-back isomorphism $L^* : |W|^\alpha \to |V|^\alpha$ and a push-forward isomorphism $L_* = (L^{-1})^* : |V|^\alpha \to |W|^\alpha$.*

Proof. (1) From definition we immediately see that $|V|^\alpha$ is a vector space. Since $|\omega|^\alpha \in |V|^\alpha$ for any $\omega \in \Lambda^n(V)$, $|V|^\alpha$ is at least one dimensional. To show $|V|^\alpha$ is exactly one dimensional, we only need to notice the transitivity of the $GL(n)$-action on $\mathcal{F}(V)$: let's suppose $\mu_1(e_1, \cdots, e_n) = \mu_2(e_1, \cdots, e_n)$ for some basis $\{e_1, \cdots, e_n\}$ of V. Then for any $v_1, \cdots, v_n$, one can choose a unique linear map A on V that sends e_i to v_i. It follows that

$$\begin{aligned}\mu_1(v_1, \cdots, v_n) &= |\det A|^\alpha \ \mu_1(e_1, \cdots, e_n) \\ &= |\det A|^\alpha \ \mu_2(e_1, \cdots, e_n) = \mu_2(v_1, \cdots, v_n).\end{aligned}$$

Thus $\mu_1 = \mu_2$ if they coincide on one basis.

(2) If $\rho \in |V|^\alpha$ and $\tau \in |V|^\beta$, then obviously $\rho \cdot \tau \in |V|^{\alpha+\beta}$.

(3) If $\mu \in |V|^\alpha$, then $\bar{\mu} \in |V|^{\bar{\alpha}}$ since

$$\overline{\mu(Av_1, \cdots, Av_n)} = |\det A|^{\bar{\alpha}} \ \overline{\mu(v_1, \cdots, v_n)}.$$

(4) Suppose we have two α-densities $\rho \in |V'|^\alpha$ and $\tau \in |V''|^\alpha$. We pick any basis $(e_1, \cdots, e_k)$ of V' and extend it to a basis $(e_1, \cdots, e_k, e_{k+1}, \cdots, e_n)$ of V. Then the images of $e_{k+1}, \cdots, e_n$ under the map $V \to V''$, denoted as $e'_{k+1}, \cdots, e'_n$, is a basis of V''. Now we define an α-density μ on V via

$$\mu(e_1, \cdots, e_n) = \rho(e_1, \cdots, e_k)\tau(e'_{k+1}, \cdots, e'_n)$$

(and extend to other bases via linear transfomation). We have to argue that the density μ defined by this way is canonical, namely, it is independent of the choice of $e_1, \cdots, e_k$ and $e_{k+1}, \cdots, e_n$. In fact, any two bases of V of this type is related by a matrix $A \in \mathrm{GL}(n)$ of the form

$$A = \begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix}.$$

Since $\det A = \det A' \det A''$, the independence of choices of bases follows and thus we get a canonical isomorphism $|V|^\alpha \simeq |V'|^\alpha \otimes |V''|^\alpha$.

(5) By definition of dual: If μ is an α-density on V, then we define

$$\mu^*(v_1^*, \cdots, v_n^*) := \mu(v_1, \cdots, v_n),$$

where $v_1, \cdots, v_n$ is the dual basis of $v_1^*, \cdots, v_n^*$. It is routine to check μ^* is a $(-\alpha)$-density on V^*: The dual basis of $Av_1^*, \cdots, Av_n^*$ is $(A^{-1})^T v_1, \cdots, (A^{-1})^T v_n$. So by our definition,

$$\begin{aligned}\mu^*(Av_1^*, \cdots, Av_n^*) &= \mu((A^{-1})^T v_1, \cdots, (A^{-1})^T v_n) \\ &= |\det(A^{-1})^T|^\alpha \mu(v_1, \cdots, v_n) \\ &= |\det A|^{-\alpha} \mu(v_1, \cdots, v_n).\end{aligned}$$

(6) The pull-back isomorphism L^* is defined to be

$$L^*\mu(v_1, \cdots, v_n) := \mu(Lv_1, \cdots, Lv_n).$$

It is an α-density since

$$\begin{aligned}\mu(LAv_1, \cdots, LAv_n) &= \mu(LAL^{-1}Lv_1, \cdots, LAL^{-1}Lv_n) \\ &= |\det A|^\alpha \mu(Lv_1, \cdots, Lv_n).\end{aligned}$$

□

¶ Densities on smooth manifolds.

For any real vector bundle $E \to X$, where X is a smooth manifold, one can consider the complex line bundle

$$|E|^\alpha \to X$$

whose fiber at x is $|E_x|^\alpha$. (Exercise: Please figure out the details of the construction of the line bundle.)

Definition 1.3. A smooth section of $|TX|^\alpha$ is called an α-density on X. We denote the set of all smooth α-densities on X as $\Gamma^\infty(|TX|^\alpha)$.

Example. The Riemannian α-density $\mu_g = \left(\sqrt{|\det(g)|}|dx_1 \wedge \cdots \wedge dx_n|\right)^\alpha$.

We can pull back densities as follows: If $f : X \to Y$ is a diffeomorphism, and μ is a density on Y, then $f^*\mu$, (the *pull-back* of μ), is a density on X defined by

$$(f^*\mu)_m(v_1, \cdots, v_n) = \mu_{f(m)}(df_m(v_1), \cdots, df_m(v_n)).$$

Other operations like multiplication, complex conjugation etc in the linear theory can also be easily extended to this setting, the only difference being: vector spaces isomorphisms gets replaced by line bundle isomorphisms. (Exercise: Try to write down the details.)

¶ Integrating 1-Densities on smooth manifolds.

Suppose $(U, x_1, \cdots, x_n)$ is a coordinate patch near $x \in X$, then we can write any 1-density on U as

$$\mu(x) = f(x)|dx_1 \wedge \cdots \wedge dx_n|$$

for some smooth function f on U. As in the case of differential forms, one can integrate a 1-density on a smooth manifolds: one first define the integral of one densities compactly supported in one coordinate charts, then extend the definition to more general one densities via partition of unity. More precisely:

Step 1. First suppose μ is a compactly supported continuous density on $\mathbb{R}^n$. Then we can write $\mu = f|dx_1 \wedge \cdots \wedge dx_n|$ for some continuous function f support on a compact set $D \subset \mathbb{R}^n$. Define

$$\int_{\mathbb{R}^n} \mu := \int_{\mathbb{R}^n} f(x)dx_1 \cdots dx_n = \int_D f(x)dx_1 \cdots dx_n.$$

To define the integration of densities on manifolds, we need the following

Lemma 1.4. *Suppose U, V are open sets in $\mathbb{R}^n$, and $\varphi : U \to V$ is a diffeomorphism, μ is a density on V, then*

$$\int_V \mu = \int_U \varphi^* \mu. \tag{2}$$

Proof. Denote $\mu = f|dx_1 \wedge \cdots \wedge dx_n|$, then

$$\varphi^*\mu = f(\varphi(x))\,|\det d\varphi|\,|dx_1 \wedge \cdots \wedge dx_n|,$$

and the lemma follows from the change of variable formula in calculus. □

Step 2. Secondly suppose μ is a 1-density on M supported on a coordinate chart (φ, U, V), we define

$$\int_U \mu := \int_V (\varphi^{-1})^*\mu.$$

This is well-defined, since if $(\widetilde{\varphi}, \widetilde{U}, \widetilde{V})$ is another coordinate chart and μ is also supported in $\widetilde{U}$, then

$$\int_{\widetilde{V}} (\widetilde{\varphi}^{-1})^*\mu = \int_V (\widetilde{\varphi} \circ \varphi^{-1})^*(\widetilde{\varphi}^{-1})^*\mu = \int_V (\varphi^{-1})^*\mu,$$

where we used the fact that $\widetilde{\varphi} \circ \varphi^{-1}$ is a diffeomorphism from $\varphi(U \cap \widetilde{U})$ to $\widetilde{\varphi}(U \cap \widetilde{U})$, and that $(\widetilde{\varphi} \circ \varphi^{-1})^* = (\varphi^{-1})^* \circ \widetilde{\varphi}^*$.

Step 3. Finally suppose μ is any compactly supported continuous density on M. Take a finite open cover $\{U_i\}$ of support of μ by coordinate charts, then $\{U_i, U_0 = M - \cup U_i\}$ is a finite cover of M. The partition of unity theorem claims that there exists smooth functions ψ_i supported in U_i satisfying $0 \le \psi_i \le 1$ and $\sum \psi_i \equiv 1$. Now we can define

$$\int_M \mu = \sum \int_{U_i} \psi_i \mu.$$

It is not hard to check that this is independent of choices of open cover, and choices of partition of unity, so the integration of compactly supported densities are well defined.

The integration of densities satisfies the following propositions:

Proposition 1.5. *Let μ, ν be compactly supported densities on M.*

(1) (Linearity) $\int_M (a\mu + b\nu) = a\int_M \mu + b\int_M \nu$.
(2) (Positivity) If μ is a positive density[1], $\int_M \mu > 0$.
(3) (Invariance) If $\varphi : N \to M$ is a diffeomorphism, then $\int_M \mu = \int_N \varphi^*\mu$.

[1] A density is *positive* if it takes value in $[0, +\infty)$ and is not identically zero.

Note that if X is compact, then the set of half-densities $\Gamma^\infty(|TX|^{1/2})$ form a pre-Hilbert space if we define the inner product to be

$$\langle \rho, \tau \rangle := \int_X \rho \bar{\tau}.$$

This is the first advantage of densities: they form intrinsic Hilbert spaces; we don't need extra structures like Riemannian structure to define integrals and turn some space of functions into a Hilbert space.

¶ Push-forward under a fibration.

Using the integral of densities, we can also push-forward a half-density along a fibration. More precisely, suppose $\pi : Z \to X$ is a fibration with compact fibers. Denote by $F_x = \pi^{-1}(x)$ the fiber over x. Then for any $z \in F_x$, we have an exact sequence of vector spaces

$$0 \longrightarrow T_z F_x \longrightarrow T_z X \xrightarrow{d\pi_z} T_x X \longrightarrow 0$$

which gives an isomorphism between the space of 1-densities

$$|T_z F_x| \otimes |T_x X| \simeq |T_z Z|. \tag{3}$$

Now let μ be a one density on Z. We first fix a one density ν on X. Then according to the isomorphism above we get a one density σ on F_x so that $\sigma \otimes \nu = \mu$. We define the *push-forward* of μ under the fibration π to be the one density defined pointwise via

$$\pi_*(\mu) := (\int_{F_x} \sigma)\nu.$$

Note that if we replace ν by $c\nu$, then σ is replaced by $\frac{1}{c}\sigma$, where $c = c(x)$ is a constant on the fiber F_x, so the push-forward is well defined.

Locally if $(x_1, \cdots, x_n, s_1, \cdots, s_d)$ are coordinates on Z, with $(x_1, \cdots, x_n)$ coordinates on X, and if

$$\mu = u(x_1, \cdots, x_n, s_1, \cdots, s_d)|dx_1 \cdots dx_n ds_1 \cdots ds_d|$$

is compactly supported in one chart, then

$$\pi_* \mu = \left(\int u(x_1, \cdots, x_n, s_1, \cdots, s_d) ds_1 \cdots ds_d \right) |dx_1 \cdots dx_n|. \tag{4}$$

¶ Pseudodifferential operators acting on half densities.

With densities at hand, we can develop an intrinsic theory of semiclassical pseudodifferential operators on manifolds without using Riemannian structure. [Note: from the physics point of view, the classical mechanics is described via the symplectic geometry of the phase space. We don't really need Riemannian structure to develop the theory.]

- Given any half-density $K \in \Gamma^\infty(|T(X \times Y)|^{1/2})$ (we may assume that K is compactly supported or has "Schwartz coefficients" first, and then extend to half-densities with "distributional coefficient" by duality), we can define an operator A_K mapping half-densities in $\Gamma^\infty(|TX|^{1/2})$ to half-densities in $\Gamma^\infty(|TY|^{1/2})$ whose Schwartz kernel is K:

$$A\mu := \int_X K\mu.$$

- For any $a \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$, we can define a half-density

$$K_a(x,y)|dx|^{1/2}|dy|^{1/2} = \left(\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi} a(\frac{x+y}{2},\xi)d\xi\right)|dx|^{1/2}|dy|^{1/2}$$

  and then define the Weyl quantization $\widehat{a}^W$ of a to be the operator acting on half-densities whose Schwartz kernel is the half-density $K_a(x,y)|dx|^{1/2}|dy|^{1/2}$:

$$\widehat{a}^W(u(x)|dx|^{1/2}) = \left(\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}(x-y)\cdot\xi} a(\frac{x+y}{2},\xi)u(y)d\xi dy\right)|dx|^{1/2}.$$

- By using invariant symbols a, again we can define pseudodifferential operators on compact manifolds as in Lecture 20. Recall that in Lecture 20, a crucial step is to prove that under a diffeomorphism f, the operator $(f^{-1})^*\widehat{a}^W f^*$ is again a pseudodifferential operator with invariant symbol of the form $\widetilde{f}^*a + O(\hbar)$. For pseudodifferential operators acting on half-densities, one can show when acting on half-densities, the operator $(f^{-1})^*\widehat{a}^W f^*$ is again a pseudodifferential operator acting on half-densities with invariant symbol of the form $\widetilde{f}^*a + O(\hbar^2)$. (For details of the proof, c.f. Zworski, §9.2-9.3).
- Another advantage of considering pseudodifferential operators as operators acting on half-densities lies in the following fact: one can define a conception of *sub-principal symbol* for pseudodifferential operators acting on half-densities on manifolds.

2. The Enhanced symplectic "category"

Similarly semiclassical Fourier integral operators should also be defined on the space of half-densities. To give such a global (coordinate free) definition, we will enhance the "symplectic category" $\mathcal{S}$ by adding half densities as a piece of data on canonical relations, so that integrals are intrinsically defined. More precisely, we would like to define a "category" $\mathcal{ES}$ with

- Objects = symplectic manifolds
- $\mathrm{Mor}(M_1, M_2)$ = pairs (Γ, σ), where $\Gamma \subset M_1 \times M_2^-$ is a Lagrangian submanifold, and σ is a half density on Γ.

Still, the question is: Given two morphisms $(\Gamma_i, \sigma_i) \in \mathrm{Mor}(M_i, M_{i+1})$, how do we compose? We have seen how to compose canonical relations modulo transversal/clean intersection conditions. So the remaining question is: Given half densities σ_i on Γ_i, how to form a new half density $\sigma_2 \circ \sigma_1$ on $\Gamma_2 \circ \Gamma_1$?

¶ Some linear theory.

Let V_1, V_2 and V_3 be symplectic vector spaces, and $\Gamma_1 \subset V_1 \times V_2^-$, $\Gamma_2 \subset V_2 \times V_3^-$ linear canonical relations. Let

$$\pi : \Gamma_1 \to V_2, (v_1, v_2) \mapsto v_2$$

and

$$\rho : \Gamma_2 \to V_2, (v_2, v_3) \mapsto v_2$$

be the canonical projections onto V_2. Since Γ_1, Γ_2 are Lagrangian subspaces, one can easily check

$$(\mathrm{Im}(\pi))^{\Omega_2} = \{v \in V_2 \mid (0, v) \in \Gamma_1\}$$

and

$$(\mathrm{Im}(\rho))^{\Omega_2} = \{v \in V_2 \mid (v, 0) \in \Gamma_2\},$$

where Ω_2 is the symplectic form on V_2. Let F be the *fiber product* of π and ρ, namely

$$\begin{aligned} F &= \{(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \mid \pi(\gamma_1) = \rho(\gamma_2)\} \\ &= \{(v_1, v_2, v_3) \mid (v_1, v_2) \in \Gamma_1, (v_2, v_3) \in \Gamma_2\}. \end{aligned}$$

Let $\tau : \Gamma_1 \times \Gamma_2 \to V_2$ be the map

$$\tau(\gamma_1, \gamma_2) = \pi(\gamma_1) - \rho(\gamma_2). \tag{5}$$

Then we have a short exact sequence

$$0 \to F \hookrightarrow \Gamma_1 \times \Gamma_2 \xrightarrow{\tau} \mathrm{Im}(\tau) \to 0$$

and thus a canonical isomorphism

$$|F|^{\frac{1}{2}} \otimes |\mathrm{Im}(\tau)|^{\frac{1}{2}} \simeq |\Gamma_1|^{\frac{1}{2}} \otimes |\Gamma_2|^{\frac{1}{2}}. \tag{6}$$

If we let α be the map

$$\alpha : F \to V_1 \times V_3, \quad (v_1, v_2, v_3) \to (v_1, v_3),$$

then by definition $\mathrm{Im}(\alpha) = \Gamma_2 \circ \Gamma_1$ so we have a short exact sequence

$$0 \to \ker(\alpha) \hookrightarrow F \to \Gamma_2 \circ \Gamma_1 \to 0$$

from which we get a canonical isomorphism

$$|F|^{\frac{1}{2}} \simeq |\Gamma_2 \circ \Gamma_1|^{\frac{1}{2}} \otimes |\ker(\alpha)|^{\frac{1}{2}}. \tag{7}$$

Moreover, from $\mathrm{Im}(\tau) = \mathrm{Im}(\pi) + \mathrm{Im}(\rho)$ we get

$$\mathrm{Im}(\tau)^{\Omega_2} = \mathrm{Im}(\pi)^{\Omega_2} \cap \mathrm{Im}(\rho)^{\Omega_2} = \{v \in V_2 \mid (0, v) \in \Gamma_1, (v, 0) \in \Gamma_2\} = \ker(\alpha).$$

As an consequence, we get an identification

$$V_2/\ker(\alpha) = V_2/(\mathrm{Im}(\tau))^{\Omega_2} \simeq (\mathrm{Im}(\tau))^*$$

and thus a canonical isomorphism

$$|V_2/\ker(\alpha)|^{-\frac{1}{2}} \simeq |\mathrm{Im}(\tau)|^{\frac{1}{2}}. \tag{8}$$

On the other hand side, from the short exact sequence

$$0 \to \ker(\alpha) \hookrightarrow V_2 \to V_2/\ker(\alpha) \to 0$$

we get

$$|V_2|^{\frac{1}{2}} \simeq |\ker(\alpha)|^{\frac{1}{2}} \otimes |V_2/\ker(\alpha)|^{\frac{1}{2}}.$$

Using the symplectic form (and thus the Liouville volume form) on V_2 one can identify $|V_2|^{\frac{1}{2}} \simeq \mathbb{C}$. As an consequence, we get from (8) a canonical isomorphism

$$|\ker(\alpha)|^{\frac{1}{2}} \simeq |V_2/\ker(\alpha)|^{-\frac{1}{2}} \simeq |\mathrm{Im}(\tau)|^{\frac{1}{2}}. \tag{9}$$

Combining (6), (7) and (9), we get

Theorem 2.1. *There is a canonical isomorphism*

$$|\Gamma_1|^{\frac{1}{2}} \otimes |\Gamma_2|^{\frac{1}{2}} \simeq |\ker(\alpha)| \otimes |\Gamma_2 \circ \Gamma_1|^{\frac{1}{2}}. \tag{10}$$

In particular we see that if τ is surjective, then $\ker(\alpha) = 0$ and the canonical isomorphism in the theorem becomes

$$|\Gamma_1|^{\frac{1}{2}} \otimes |\Gamma_2|^{\frac{1}{2}} \simeq |\Gamma_2 \circ \Gamma_1|^{\frac{1}{2}}.$$

In other words, given half densities σ_i on Γ_i one can canonically obtain a half density $\sigma_2 \circ \sigma_1$ on $\Gamma_2 \circ \Gamma_1$.

¶ Then enhanced symplectic "category".

Now suppose M_i, $i = 1, 2, 3$ are symplectic manifolds, and $\Gamma_i \subset M_i \times M_{i+1}^-$ canonical relations. We will assume as before that $\pi : \Gamma_1 \to M_2$ and $\rho : \Gamma_2 \to M_2$ intersect transversally (or cleanly), so that the fiber product

$$F = \{(m_1, m_2, m_3) \mid (m_1, m_2) \in \Gamma_1, (m_2, m_3) \in \Gamma_2\}$$

is a manifold, and the tangent space of F at $m = (m_1, m_2, m_3)$ is

$$T_m F = \{(v_1, v_2, v_3) \mid v_i \in T_{m_i} M_i, (v_i, v_{i+1}) \in T_{(m_i, m_{i+1})} \Gamma_i\}.$$

As in the linear case we define the map

$$\alpha : F \to M_1 \times M_3, \quad (m_1, m_2, m_3) \mapsto (m_1, m_3).$$

The transversal/clean intersection condition implies that α is a constant rank map and that $\Gamma_2 \circ \Gamma_1$ is an immersed canonical relation. We will further assume

- α is proper,
- the level sets of α are connected,

then $\Gamma_2 \circ \Gamma_1$ is an embedded Lagrangian submanifold of $M_1 \times M_3$, and

$$\alpha : F \to \Gamma_2 \circ \Gamma_1 \tag{11}$$

is a fibration with compact fibers.

For any point $m \in F$ we denote $q = \alpha(m) \in \Gamma_2 \circ \Gamma_1$. Then the fiber $F_q = \alpha^{-1}(q)$ is compact and $m \in F_q$. Moreover, by definition

$$T_m F_q = \ker(d\alpha_m).$$

According to Theorem 2.1, we get a canonical identification

$$|T_m F_q| \otimes |T_q(\Gamma_2 \circ \Gamma_1)|^{\frac{1}{2}} \simeq |T_{(m_1,m_2)}\Gamma_1|^{\frac{1}{2}} \otimes |T_{(m_2,m_3)}\Gamma_2|^{\frac{1}{2}}.$$

Fix any non-zero half density $\sigma_q \in |T_q(\Gamma_2 \circ \Gamma_1)|^{\frac{1}{2}}$. Let σ_1, σ_2 be half densities on Γ_1 and Γ_2 respectively. By the identification above, there is a unique one density $\nu_m \in |T_m F_q|$ which depends on m smoothly so that

$$\nu_m \otimes \sigma_q = \sigma_1^{(m_1,m_2)} \otimes \sigma_2^{(m_2,m_3)}.$$

Since F_q is compact, we can integrate ν over F_q.

Definition 2.2. We define the composition of σ_1 and σ_2 to be

$$(\sigma_2 \circ \sigma_1)(q) = (\int_{F_q} \nu)\sigma_q. \tag{12}$$

Note: if we change σ_q to $c\sigma_q$, then ν is changed to $\frac{1}{c}\nu$, so the right hand side of (12) is independent of the choice of σ_q and gives a well-defined half-density on $\Gamma_2 \circ \Gamma_1$.

Now we can describe the enhanced symplectic "category" $\mathcal{ES}$:

- Objects=symplectic manifolds
- Mor(M_1, M_2)= pairs (Γ, σ), where Γ is a canonical relation from M_1 to M_2, and $\sigma \in \Gamma^\infty(|T\Gamma|^{\frac{1}{2}})$ is a half-density on Γ.

The composition of morphisms is given by

$$(\Gamma_2, \sigma_2) \circ (\Gamma_1, \sigma_1) = (\Gamma_2 \circ \Gamma_1, \sigma_2 \circ \sigma_1). \tag{13}$$

We will leave the associativity of the composition as an exercise.

Example. If $M_2 = M_3$, $\Gamma_2 = \Delta_{M_2}$ is the diagonal (which could be identified with M_2), and $\sigma_2 = \sigma_\Delta = \sqrt{|\omega_2^n/n!|}$ is the canonical half density that corresponds to the symplectic volume form on M_2, then for any (Γ_1, σ_1), one has

$$(\Delta_{M_2}, \sigma_\Delta) \circ (\Gamma_1, \sigma_1) = (\Gamma_1, \sigma_1).$$

Similarly one has

$$(\Gamma_2, \sigma_2) \circ (\Delta_{M_1}, \sigma_\Delta) = (\Gamma_2, \sigma_2).$$