LECTURE 29-30 FIO – SEMICLASSICAL FIOs

1. Generating functions with respect to a fibration

¶ Recall: Generating functions of a horizontal Lagrangian.

Let $M = T^*X$ be the cotangent bundle of a smooth manifold X. Recall

- A horizontal submanifold (=the graph of a 1-form μ)
$$\Lambda_\mu = \{(x, \mu_x) \mid x \in X\}$$
is a Lagrangian submanifold of M if and only if $d\mu = 0$.
- If $\mu = d\varphi$ is exact, then we call $\varphi \in C^\infty(X)$ a generating function of Λ_μ.

For example, if we take $X = \mathbb{R}^n_x \times \mathbb{R}^n_y$, then
$$\varphi(x, y) = -x \cdot y$$
is a generating function of the Lagrangian submanifold
$$\Lambda = \{(x, y, \xi, \eta) \mid \xi = -y, \eta = -x\}.$$
Note that $\Lambda = \sigma_2 \circ G_F$ is the "twisting" of the graph of the symplectomorphism
$$F : T^*\mathbb{R}^n_x \to T^*\mathbb{R}^n_y, \quad (x, \xi) \mapsto (-\xi, x).$$

¶ Generating function with respect to a fibration.

Unfortunately not all Lagrangian submanifolds are generated (even locally) by those kind of generating functions: there are many interesting non-horizontal Lagrangian submanifolds. For example, any smooth map $f : X \to Y$ "lifts" to a canonical relation (which generalize the naturality of the cotangent bundle: any diffeomorphism lifts to a symplectomorphism between cotangent bundles)
$$\Gamma_f := \sigma_2(N^*G_f) = \{(x, y, \xi, \eta) \mid y = f(x), \xi = (df_x)^T\eta\}.^1$$
In what follows we will extend the conception of generating functions by introducing "auxiliary variables" (to "separate the non-horizontal directions) so that every Lagrangian submanifold of T^*X *is* locally represented by such a generating function.

Let Z, X are smooth manifolds and $\pi : Z \to X$ a smooth fibration. Then
$$\Gamma_\pi = \{(z, \zeta, x, \xi) \mid x = \pi(z), \zeta = (d\pi_z)^T\xi\}$$
is a canonical relation in $T^*Z \times (T^*X)^-$. Let Λ_φ be a horizontal Lagrangian submanifold of T^*Z generated by a function $\varphi \in C^\infty(Z)$, i.e.
$$\Lambda_\varphi = \{(z, d\varphi(z)) \mid z \in Z\}.$$

[1] Check this expression!

Then one can think of Λ_φ as a morphism from "*pt*" to T^*Z. So if Γ_π and Λ_φ are transversally composable,[2] then

$$\begin{aligned}\Lambda := \Gamma_\pi \circ \Lambda_\varphi &= \{(x,\xi) \mid \exists(z,\zeta,x,\xi) \in \Gamma_\pi, \exists(z,\zeta) \in \Lambda_\varphi\} \\ &= \{(x,\xi) \mid x = \pi(z) \text{ and } d\varphi_z = (d\pi_z)^T\xi\}.\end{aligned} \tag{1}$$

is a canonical relation from "*pt*" to T^*X, i.e. a Lagrangian submanifold of T^*X.

Definition 1.1. We call $\varphi \in C^\infty(Z)$ a *generating function* of $\Lambda \subset T^*X$ with respect to the fibration $\pi : Z \to X$.

¶ Consequence of transversality.

Next let's look for conditions so that Γ_π and Λ_φ are transversally composable. Let H^*Z be the horizontal subbundle of T^*Z which is the image of Γ_π under the projection $\rho : \Gamma_\pi \hookrightarrow T^*Z \times T^*X \to T^*Z$. In other words, the fiber of H^*Z at z is

$$(H^*Z)_z = \{(d\pi_z)^T\xi \mid \xi \in T^*_{\pi(z)}X\}.$$

Since H^*Z is a subbundle of T^*Z, one has a vector bundle short exact sequence

$$0 \to H^*Z \to T^*Z \to V^*Z \to 0, \tag{2}$$

where $(V^*Z)_z = T^*_zZ/(H^*Z)_z \simeq T^*_z(\pi^{-1}(\pi(z)))$ is the cotangent space to the fiber through z. From the exact sequence, any section $d\varphi$ of T^*Z gives a section $d_{vert}\varphi$ of V^*Z, and H^*Z gets projected to the zero section of V^*Z.

Note the transversality condition of Γ_π and Λ_φ now becomes

$$\begin{aligned}&\pi : \Lambda_\varphi \to T^*Z \text{ intersect } \rho : \Gamma_\pi \to T^*Z \text{ transversally} \\ \Longleftrightarrow &\Lambda_\varphi \text{ intersect } \rho : \Gamma_\pi \to T^*Z \text{ transversally in } T^*Z \\ \Longleftrightarrow &\Lambda_\varphi \text{ intersect } H^*Z \text{ transversally in } T^*Z \\ \Longleftrightarrow &d_{vert}\varphi \text{ intersect the zero section } Z \text{ transversally in } V^*Z\end{aligned}$$

It follows that under the transversal intersection assumption, the intersection

$$C_\varphi := \{z \in Z \mid (d_{vert}\varphi)_z = 0\} \tag{3}$$

is a submanifold of Z whose dimension is

$$\dim C_\varphi = \dim Z + \dim Z - \dim V^*Z = \dim X.$$

Furthermore, the short exact sequence also implies that at any $z \in C_\varphi$,

$$d\varphi_z = (d\pi_z)^T\xi$$

for a unique $\xi \in T^*_{\pi(z)}X$, and by (1), $\Lambda = \Gamma_\pi \circ \Lambda_\varphi$ is the image of the map

$$C_\varphi \to T^*X, \quad z \mapsto (\pi(z), \xi).$$

[2]Recall from Lecture 27 that two canonical relations are transversally composable if the maps π and ρ intersect transversally, which implies that the map $\alpha = \kappa \circ \iota$ is of constant rank; moreover we assume $\kappa \circ \iota$ is proper with connected fiber.

We will denote this map by p_φ:

$$p_\varphi : C_\varphi \to \Lambda. \tag{4}$$

¶ The generating function in local coordinates.

Locally assume X is an open subset of $\mathbb{R}^n$ and $Z = X \times \mathbb{R}^k$. Let (x, s) be the coordinates on Z so that $\varphi = \varphi(x, s)$. Then $C_\varphi \subset Z$ is defined by the k equations

$$\frac{\partial \varphi}{\partial s_i} = 0, \qquad i = 1, 2, \cdots k, \tag{5}$$

and the transversality condition becomes

Transversality Assumption: the differentials of these functions,

$$d\left(\frac{\partial \varphi}{\partial s_i}\right), \qquad i = 1, 2, \cdots, k$$

are linearly independent.

In this case, $\Lambda \subset T^*X$ is the image of the embedding

$$C_\varphi \to T^*X, \quad (x, s) \mapsto (x, d_x\varphi(x, s)).$$

Example. Let $Y \subset X$ be a submanifold defined by k equations

$$f_1(x) = \cdots = f_k(x) = 0$$

and assume that these equations are functionally independent, i.e. $df_1, \cdots, df_k$ are linearly independent. Let $\varphi : X \times \mathbb{R}^k \to \mathbb{R}$ be the function

$$\varphi(x, s) = \sum f_i(x) s_i. \tag{6}$$

We claim that $\Lambda = \Gamma_\pi \circ \Lambda_\phi$ is the conormal bundle N^*Y of Y. In fact, since $\frac{\partial \varphi}{\partial s_i} = f_i$ we see

$$C_\varphi = Y \times \mathbb{R}^k,$$

and the map $C_\varphi \to T^*X$ is given by

$$(x, s) \mapsto (x, \sum s_i d_x f_i(x)).$$

The conclusion follows since $d_x f_i$'s span the conormal fiber to Y at each x.

Example. In particular, if we let $X = \mathbb{R}^n \times \mathbb{R}^n$ and let Y be the diagonal

$$Y = \text{diag}(X) = \{(x, x) \mid x \in X\},$$

then $Y \subset X$ is defined by the equations

$$x_i - y_i = 0, \qquad i = 1, 2, \cdots, n.$$

So the function

$$\varphi(x, y, s) = (x - y) \cdot s = \sum_i (x_i - y_i) s_i, \tag{7}$$

is the generating function of $N^*(\text{diag}(X))$.

¶ General facts about the generating function.

Of course one may ask: Given any Lagrangian submanifold $\Lambda \subset T^*X$, does there exist any fibration $\pi : Z \to X$ and $\varphi \in C^\infty(Z)$ so that φ is a generating function of Λ? If yes, is it unique? We state without proof the following general results. For details, c.f. Guillemin-Sternberg §5.9 and §5.11:

Theorem 1.2 (Existence)**.** *For any Lagrangian submanifold $\Lambda \subset T^*X$ and any $p \in \Lambda$, there exist a fibration $\pi : Z \to X$ and a smooth function $\varphi \in C^\infty(Z)$ so that φ is a generating function of Λ near p.*

Theorem 1.3. *(Uniqueness up to "Hörmander moves") Suppose φ_i, $i = 1, 2$, are generating functions for the same Lagrangian submanifold $\Lambda \subset T^*X$ with respect to fibrations $\pi_i : Z_i \to X$. Then locally one can obtain one description from the other by applying a sequence of "moves" of the following three types:*

(1) *Adding a constant: replace φ by $\varphi + c$.*
(2) *Equivalence: For a diffeomorphism $g : Z \to \widetilde{Z}$, replace (π, φ) by $(g^*\pi, g^*\varphi)$.*
(3) *Increasing the number of fiber variables: replace Z by $Z = Z \times \mathbb{R}^d$ and φ by $\varphi(z) + \frac{1}{2}\langle Az, z\rangle$, where A is a non-degenerate $d \times d$ matrix.*

In Guillemin-Sternberg Chapter 5, many nice facts were proven for the generating functions (with respect to fibrations). We list several of them without proof:

- If $\Gamma \in \mathrm{Mor}(T^*X, T^*Y)$ is a canonical relation, $\pi : Z \to X \times Y$ a fibration, and φ a generating function of Γ with respect to this fibration. Suppose locally $\varphi = \varphi(x, y, s)$. Then the function $\psi(y, x, s) = -\varphi(x, y, s)$ is a generating function for the transpose canonical relation
$$\Gamma^T = \{(y, \eta, x, \xi) | (x, \xi, y, \eta) \in \Gamma\} \in \mathrm{Mor}(T^*Y, T^*X).$$
- If $\Gamma_i \in \mathrm{Mor}(T^*X_i, T^*X_{i+1})$, $i = 1, 2$ are canonical relations which are transversally composable, $\pi_i : Z_i \to X_i \times X_{i+1}$ are fibrations and $\varphi_i \in C^\infty(Z_i)$ are generating functions for Γ_i with respect to π_i, then one can construct a fibration $Z \to X_1 \times X_3$ with
$$Z = (\pi_1 \times \pi_2)^{-1}(X_1 \times \Delta_{X_2} \times X_3), \tag{8}$$
Let φ be the restriction to Z of the function
$$(z_1, z_2) \mapsto \varphi_1(z_1) + \varphi_2(z_2), \tag{9}$$
then φ is a generating function for $\Gamma_2 \circ \Gamma_1$ with respect to the fibration $Z \to X_1 \times X_3$.
- Suppose that the fibration $\pi : Z \to X$ can be factored as a succession of fibrations $\pi = \pi_1 \circ \pi_0$, where $\pi_0 : Z \to Z_1$ and $\pi_1 : Z_1 \to X$ are fibrations. Moreover, suppose that the restriction of the generating function φ to each fiber $\pi_0^{-1}(z_1)$ has a unique non-degenerate critical point $\gamma(z_1)$, so that we get a section $\gamma : Z_1 \to Z$. Then the function $\phi_1 = \gamma_1^*\phi$ is a generating function of Λ with respect to π_1.

2. Oscillatory half densities

¶ Bohr-Sommerfeld conditions.

Now assume X is a smooth manifold, $\Lambda \subset T^*X$ a Lagrangian submanifold. Let $\varphi \in C^\infty(Z)$ be a (global!) generating function for Λ with respect to a fibration $\pi : Z \to X$. In developing the global theory, we need to assume that Λ satisfies the following *Bohr-Sommerfeld* condition:

In what follows, we will assume that Λ is *exact* in the sense that

$$\iota_\Lambda^* \alpha_{T^*X} = d\varphi_\Lambda \tag{10}$$

for some $\varphi_\Lambda \in C^\infty(\Lambda)$, where α_{T^*X} is the canonical 1 form on T^*X.

One major application of the Bohr-Sommerfeld assumption on Λ is to fix the arbitrary constant in the generating function, which need to be kept tract of in applications. Let $\iota : C_\varphi \hookrightarrow Z$ be the inclusion and $p_\varphi : C_\varphi \to \Lambda$ be the map (4).

Lemma 2.1. *$d(\iota^*\varphi - p_\varphi^*\varphi_\Lambda) = 0$.*

Proof. In fact, by definition of phase function φ_Λ,

$$d(\iota^*\varphi - p_\varphi^*\varphi_\Lambda) = \iota^* d\varphi - (\iota_\Lambda \circ p_\varphi)^*\alpha_{T^*X},$$

As we have seen, in local coordinates $Z = X \times S \subset \mathbb{R}^n \times \mathbb{R}^k$, then

$$C_\varphi = \{(x,s) \mid \frac{\partial \varphi}{\partial s_i}(x,s) = 0, 1 \le i \le k\},$$

and the map $p_\varphi : C_\varphi \to \Lambda$ is the map

$$p_\varphi(x,s) = (x, \frac{\partial\varphi}{\partial x}(x,s)).$$

It follows

$$\iota^* d\varphi = \iota^*(\sum \frac{\partial\varphi}{\partial x_i}dx_i + \frac{\partial\varphi}{\partial s_i}ds_i) = \sum \frac{\partial\varphi}{\partial x_i}dx_i.$$

On the other hand, since $\iota_\Lambda \circ p_\varphi(x,s) = (x, \frac{\partial\varphi}{\partial x})$,

$$(\iota_\Lambda \circ p_\varphi)^*\alpha_{T^*X} = (\iota_\Lambda \circ p_\varphi)^* \sum \xi_i dx_i = \sum \frac{\partial\varphi}{\partial x_i}dx_i.$$

□

In what follows, we will fix a choice of such an *exact phase function* φ_Λ, and we will fix the constant in the generating function φ by requiring

$$\iota^*\varphi = p_\varphi^*\varphi_\Lambda. \tag{11}$$

¶ Oscillatory half densities.

Let $d = \dim Z - \dim X$ be the fiber dimension. For any $k \in \mathbb{Z}$, we define $I_0^k(X, \Lambda)$, the space of compactly supported oscillatory half densities on X associated with Λ, to be

$$I_0^k(X,\Lambda) = \{\mu = \hbar^{k-\frac{d}{2}} \pi_*(a(z,\hbar) e^{i\frac{\varphi(z)}{\hbar}} \tau) \mid a \in C_0^\infty(Z \times \mathbb{R})\}, \tag{12}$$

where τ is a nowhere vanishing half-density on Z. (Obviously the space is independent of the choice of τ.) Similarly we define $I^k(X, \Lambda)$, the space of oscillatory half densities on X associated with Λ, to be the set consists of those half densities μ so that $\rho\mu \in I_0^k(X, \Lambda)$ for all $\rho \in C_0^\infty(X)$.

Locally we may assume $Z = X \times S$, where S is an open set in $\mathbb{R}^d$. We may choose our fiber half-density to be the Euclidean one $ds^{\frac{1}{2}}$ and choose τ to be $\tau_0 \otimes ds^{\frac{1}{2}}$ with τ_0 a nowhere vanishing half-density on X. Then $\mu \in I_0^k(X, \Lambda)$ is of the form

$$\hbar^{k-\frac{d}{2}} \left(\int_S a(x,s,\hbar) e^{\frac{i}{\hbar}\varphi(x,s)} ds \right) \tau_0.$$

¶ Independence of generating function.

We must show that the above definition is also independent of the choices of generating functions. Let $\pi : Z_i \to X$, $i = 1, 2$ be two fibrations, and φ_i be a generating function of Λ with respect to π_i.

It is enough to do this locally. Recall that the two generating functions φ_1 and φ_2 are related by

(a) Replace φ by $\varphi + c$.
(b) For a diffeomorphism $g : Z \to \tilde{Z}$, replace π by $g^*\pi$ and φ by $g^*\varphi$.
(c) Replace Z by $Z = Z \times \mathbb{R}^d$ and φ by $\varphi(z) + \frac{1}{2}\langle Az, z\rangle$, where A is a non-degenerate $d \times d$ matrix.

We have already get rid of type (a) by requiring Λ to satisfy the Bohr-Sommerfeld condition (10) and fixing the constant in the generating function via the normalization condition (11). If two densities are related by a type (b) change, then by a change of variable argument it is not hard to prove

$$(\pi_2)_*(ae^{\frac{i}{\hbar}\varphi_2} g_*\tau_1) = (\pi_1)_*(g^* a e^{\frac{i}{\hbar}\varphi_1} \tau_1)$$

so the spaces defined via φ_1 and via φ_2 are the same.

Now suppose φ_1 and φ_2 are related by a type (c) change. Without loss of generality, we may assume $Z_2 = Z_1 \times S$, where S is an open subset of $\mathbb{R}^m$, and

$$\varphi_2(z,s) = \varphi_1(z) + \frac{1}{2} s^T A s,$$

where A is a symmetric non-degenerate $m \times m$ matrix. Let d be the fiber dimension of $Z_1 \to X$, then the fiber dimension of $Z_2 \to X$ is $d + m$. Let τ_1 be a nowhere vanishing half density on Z_1, then $\tau_1 \otimes ds^{\frac{1}{2}}$ is a nowhere vanishing half density on Z_2. Using the generating function φ_2 we get the expressions

$$\hbar^{k-\frac{d+m}{2}}(\pi_2)_* a_2(z,s,\hbar) e^{\frac{i}{\hbar}\varphi_2(z,s)} \tau_1 \otimes ds^{\frac{1}{2}}.$$

Let $\pi_{2,1} : Z_2 \to Z_1$ be the projection on to the first factor so that $(\pi_2)_* = (\pi_1)_* \circ (\pi_{2,1})_*$. Then by definition, $(\pi_{2,1})_*$ acts as

$$(\pi_{2,1})_*(a_2(z,s,\hbar)e^{\frac{i}{\hbar}\varphi_2(z,s)}\tau_1 \otimes ds^{\frac{1}{2}}) = \left(\int a_2(z,s,\hbar)e^{\frac{i}{2\hbar}s^T As} ds\right) e^{\frac{i}{\hbar}\varphi_1}\tau_1.$$

Now the conclusion follows from the lemma of stationary phase (with quadratic phase).

In conclusion, we proved

Theorem 2.2. *The space $I_0^k(X,\Lambda)$ (and thus $I^k(X,\Lambda)$) is intrinsically defined (provided Λ is exact and we fix a choice of φ_Λ on Λ).*

3. Semiclassical Fourier integral operators

¶ The definition.

Now suppose X_1, X_2 are smanifolds. We will denote $M_i = T^*X_i$, $i = 1,2$. Suppose $\Gamma \subset M_1 \times M_2^-$ is an *exact* canonical relation. Then

$$\Lambda = \sigma_2 \circ \Gamma$$

is an exact Lagrangian submanifold of T^*X, where $X = X_1 \times X_2$. Associated with Λ we have the space of compactly supported oscillatory half densities $I_0^k(X,\Lambda)$. If we fix a nowhere vanishing one density dx_1 on X_1 and a nowhere vanishing one density dx_2 on X_2, then a typical element in $I_0^k(X,\Lambda)$ is of the form

$$\mu = \hbar^{k-\frac{d}{2}}\left(\int_S a(x_1,x_2,s,\hbar)e^{\frac{i}{\hbar}\varphi(x_1,x_2,s)} ds\right) dx_1^{\frac{1}{2}} dx_2^{\frac{1}{2}}$$

With some abuse of notion we let $L^2(X_i)$ be the Hilbert space of L^2 half densities on X_i. Then associated to each $\mu = u(x_1,x_2,\hbar)dx_1^{\frac{1}{2}}dx_2^{\frac{1}{2}} \in I_0^k(X,\Lambda)$ we can define an integral operator $F_\mu : L^2(X_1) \to L^2(X_2)$ via

$$F_\mu(f dx_1^{\frac{1}{2}}) = \left(\int f(x_1)u(x_1,x_2,\hbar)dx_1\right) dx_2^{\frac{1}{2}} \tag{13}$$

Definition 3.1. Such operators are called *compactly supported semi-classical Fourier integral operators* of order $m = k + \frac{n_2}{2}$, where $n_2 = \dim X_2$. The space of these operators is denoted by $\mathcal{F}_0^m(\Gamma)$.

Remark. We could loose the conditions on u by requiring only $u(x_1,x_2,\hbar)dx_1^{\frac{1}{2}} \in L^2(X_1)$, or more generally, with distributional coefficients. In this case we drop the subscript 0.

¶ Example: Semi-classical pseudo-differential operators.

Take $X_1 = X_2 = \mathbb{R}^n$ and

$$\Gamma = \Delta_M = \text{graph of the identity} = \{(x, s, x, s)\} \subset M \times M^-,$$

where $M = T^*\mathbb{R}^n$. Then

$$\Lambda = \sigma_2 \circ \Gamma = \{(x, x, s, -s)\} \subset T^*(\mathbb{R}^n \times \mathbb{R}^n).$$

On Λ one has $\iota_\Lambda^* \alpha_{T^*X} = \sum s_i dx_i - s_i dx_i = 0$, so one take phase function $\varphi_\Lambda = 0$.

To find a generating function, one use the fibration

$$\pi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad (x, y, s) \mapsto (x, y).$$

Then by definition,

$$\Gamma_\pi = \{(x, y, s, \eta_1, \eta_2, 0, x, y, \eta_1, \eta_2)\}.$$

If we take $\varphi \in C^\infty(\mathbb{R}^n)$ to be the function

$$\varphi(x, y, s) = \sum (x_i - y_i) s_i,$$

then

$$\Lambda_\varphi = \{(x, y, s, s, -s, x - y)\}$$

and it is easy to see

$$\Lambda = \Gamma_\pi \circ \Lambda_\varphi.$$

Moreover, the set C_φ is defined by the equations $\frac{\partial \varphi}{\partial s_i} = x_i - y_i = 0$, i.e.

$$C_\varphi = \{(x, x, s)\},$$

and the map p_φ is given explicitly by

$$p_\varphi : C_\varphi \to \Lambda, \quad (x, x, s) \mapsto (x, x, s, -s).$$

So $\iota^* \varphi = 0 = p_\varphi^* \varphi_\Lambda$. In other words, φ satisfies the normalizing condition (11).

What are the semi-classical Fourier integral operators associated to Γ? By definition, $\mathcal{F}^m(\Gamma)$ consist of those operators that maps $f(x) dx^{\frac{1}{2}}$ to

$$\hbar^{m - \frac{n}{2} - \frac{n}{2}} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot s} a(x, y, s, \hbar) f(x) dx d\xi \right) dy^{\frac{1}{2}}.$$

They are the semi-classical pseudo-differential operators (of semi-classical order m) we learned earlier in this course! (We used special symbols of the form $a(x, s, \hbar)$ or $a(y, s, \hbar)$ or $a(\frac{x+y}{2}, s, \hbar)$, but we could use general symbols $a(x, y, s, \hbar)$. One can show that any general symbol corresponds to a unique left/right/Weyl symbol. c.f A. Martinez, page 37.)

In general, if $X_1 = X_2 = X$ is a smooth manifold, then the construction above gives us semi-classical pseudo-differential operators on X. In other words, $\Psi^m(X) = \mathcal{F}^m(\Delta_M)$.

¶ **Example: The semi-classical Fourier transform.**

Let $X_1 = X_2 = \mathbb{R}^n$. Let Γ be the graph of the symplectomorphism

$$J : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad (x, y) \mapsto (-y, x),$$

i.e.

$$\Gamma = \{(x, y, -y, x)\} \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n.$$

Then

$$\Lambda = \{(x, y, -y, -x)\} \subset T^*(\mathbb{R}^n \times \mathbb{R}^n).$$

Λ is exact since $\iota_\Lambda^* \alpha_{T^*X} = -\sum y_i dx_i - \sum x_i dy_i = -d(x \cdot y)$. We just choose the phase function $\varphi_\Lambda = -x \cdot y$.

We don't need a fibration to find a generating function, since Λ is already a horizontal Lagrangian, with (normalized!) generating function

$$\varphi(x, y) = -x \cdot y.$$

(So in this example $C_\varphi = X = Z$. What is the map p_φ?)

Let $\mu = e^{-\frac{i}{\hbar} x \cdot y} dx^{\frac{1}{2}} dy^{\frac{1}{2}} \in I^0(X, \Lambda)$. What is the corresponding semi-classical Fourier integral operator? By definition $\mathcal{F}_\mu$ maps any $f(x) dx^{\frac{1}{2}}$ (with $f \in C_0^\infty(\mathbb{R}^n)$ for simplicity) to

$$\left(\int_{\mathbb{R}^n} f(x) e^{-\frac{i}{\hbar} x \cdot y} dx \right) dy^{\frac{1}{2}},$$

which is the semi-classical Fourier transform $\mathcal{F}_\hbar$!

What about the inverse (semi-classical) Fourier transform? Well, repeating the previous precess one can see that $\mathcal{F}_\hbar^{-1}$ is a semi-classical Fourier integral operator associated to the graph of the symplectomorphism (which is the inverse of the previous one)

$$J^{-1} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad (x, y) \mapsto (y, -x).$$

4. The composition of $\hbar$-FIOs

¶ The composition of phase functions.

Let X_1, X_2 and X_3 be smooth manifolds, $M_i = T^*X_i$. Let $\Gamma_i : M_i \Longrightarrow M_{i+1}$ be exact Lagrangian submanifolds, with phase function φ_{Γ_i}. Suppose Γ_1 and Γ_2 are transversally composable. Recall that this implies that the map

$$\alpha : F = \{(m_1, m_2, m_3) \mid (m_i, m_{i+1}) \in \Gamma_i)\} \to M_1 \times M_3, (m_1, m_2, m_3) \mapsto (m_1, m_3)$$

is a constant rank map which maps onto $\Gamma_2 \circ \Gamma_1$. As before we assume α is proper and has connected level sets, so that $\Gamma_2 \circ \Gamma_1$ is an embedded Lagrangian submanifold.

Theorem 4.1. *$\Gamma_2 \circ \Gamma_1$ is an exact Lagrangian submanifold of $M_1 \times M_3^-$.*

Proof. We denote $\iota_i : \Gamma_i \hookrightarrow M_i \times M_{i+1}$ for $i = 1, 2$ and denote $\iota_3 : \Gamma_2 \circ \Gamma_1 \hookrightarrow M_1 \times M_3$. Let $\rho_i : F \to \Gamma_i$ and $\pi_i : F \to M_i$ be the obvious projections. Then

$$\rho_1^*(\iota_1^* \alpha_{T^*X_1 \times T^*X_2^-}) = \pi_1^* \alpha_{T^*X_1} - \pi_2^* \alpha_{T^*X_2}.$$

Similar expressions holds for $\rho_2^*(\iota_2^*\alpha_{T^*X_2 \times T^*X_3^-})$ and $\alpha^*(\iota_3^*\alpha_{T^*X_1 \times T^*X_3^-})$, which implies

$$\rho_1^*(\iota_1^*\alpha_{T^*X_1 \times T^*X_2^-}) + \rho_2^*(\iota_2^*\alpha_{T^*X_2 \times T^*X_3^-}) = \alpha^*(\iota_3^*\alpha_{T^*X_1 \times T^*X_3^-}).$$

On the other hand, by definition

$$\iota_i^*\alpha_{T^*X_i \times T^*X_{i+1}^-} = d\varphi_{\Gamma_i}$$

for $i = 1, 2$. So if we let

$$\varphi = \rho_1^*\varphi_{\Gamma_1} + \rho_2^*\varphi_{\Gamma_2} \in C^\infty(F),$$

then

$$d\varphi = \rho_1^* d\varphi_{\Gamma_1} + \rho_2^* d\varphi_{\Gamma_2} = \alpha^*(\iota_3^*\alpha_{T^*X_1 \times T^*X_3^-}).$$

For any $p \in \Gamma_2 \circ \Gamma_1$, let $F_p = \alpha^{-1}(p)$ be the connected compact fiber over p and let $\iota_p : F_p \hookrightarrow F$ be the inclusion. Then $\alpha \circ \iota_p : F_p \to \Gamma_2 \circ \Gamma_1$ is the constant map. So

$$(\alpha \circ \iota_p)^*(\iota_3^*\alpha_{T^*X_1 \times T^*X_3^-}) = 0.$$

It follows

$$d\iota_p^*\varphi = \iota_p^* d\varphi = 0.$$

Since F_p is connected, $\iota_p^*\varphi$ is constant on F_p. In other words, φ is constant on each fiber F_p. So one can find a function $\varphi_{\Gamma_2 \circ \Gamma_1} \in C^\infty(\Gamma_2 \circ \Gamma_1)$ so that

$$\varphi = \alpha^* \varphi_{\Gamma_2 \circ \Gamma_1}.$$

Thus

$$\alpha^* d\varphi_{\Gamma_2 \circ \Gamma_1} = d\varphi = \alpha^*(\iota_3^*\alpha_{T^*X_1 \times T^*X_3^-}).$$

Since $\alpha : F \to \Gamma_2 \circ \Gamma_1$ is surjective, α^* is injective. It follows

$$d\varphi_{\Gamma_2 \circ \Gamma_1} = \iota_3^*\alpha_{T^*X_1 \times T^*X_3^-},$$

i.e. φ is a phase function for $\Gamma_2 \circ \Gamma_1$. □

Remark. In the proof we have seen that the three phase functions are related by

$$d\alpha^*\varphi_{\Gamma_2\circ\Gamma_1} = d\rho_1^*\varphi_{\Gamma_1} + d\rho_2^*\varphi_{\Gamma_2}.$$

We will fix the constant in the phase function of $\Gamma_2 \circ \Gamma_1$ by requiring

$$\alpha^*\varphi_{\Gamma_2\circ\Gamma_1} = \rho_1^*\varphi_{\Gamma_1} + \rho_2^*\varphi_{\Gamma_2}. \tag{14}$$

¶ The composition of semi-classical FIOs.

Now suppose two canonical relations $\Gamma_i : M_i \Longrightarrow M_{i+1}$ are transversally composable. Then as we mentioned earlier, if $\pi_i : Z_i \to X_i \times X_{i+1}$ are fibrations and $\varphi_i \in C^\infty(Z_i)$ are generating functions for Γ_i with respect to π_i, then if we set

$$Z = (\pi_1 \times \pi_2)^{-1}(X_1 \times \Delta_{X_2} \times X_3) \subset Z_1 \times Z_2$$

and let $\pi : Z \to X_1 \times X_3$ be the obvious fibration map, then

$$\varphi(z_1, z_2) = \varphi_1(z_1) + \varphi_2(z_2)$$

is a generating function for $\Gamma_2 \circ \Gamma_1$ with respect to π. Moreover, if we normalize φ_1 and φ_2 by fixing phase functions φ_{Γ_1} and φ_{Γ_2}, then φ is normalized with respect to $\varphi_{\Gamma_2\circ\Gamma_1}$ described above.

Theorem 4.2. *If $F_i \in \mathcal{F}_0^{m_i}(\Gamma_i)$ for $i = 1, 2$, then*

$$F_2 \circ F_1 \in \mathcal{F}_0^{m_1+m_2}(\Gamma_2 \circ \Gamma_1). \tag{15}$$

Proof. By a partition of unity argument one can assume that we have fibrations

$$\pi_1 : X_1 \times X_2 \times S_1 \to X_1 \times X_2$$

and

$$\pi_2 : X_2 \times X_3 \times S_2 \to X_2 \times X_3,$$

where $S_1 \subset \mathbb{R}^{d_1}$ and $S_2 \subset \mathbb{R}^{d_2}$ are open sets. Let φ_1 and φ_2 be generating functions of Γ_1 and Γ_2 with respect to these fibrations. Then by definition, $F_2 \circ F_1$ maps $f dx_1^{\frac{1}{2}}$ to

$$\hbar^{m_1-\frac{d_1}{2}-\frac{n_2}{2}+m_2-\frac{d_2}{2}-\frac{n_3}{2}} \left(\int_{X_2}\int_{S_2}\int_{X_1}\int_{S_1} e^{\frac{i}{\hbar}(\varphi_1(x_1,x_2,s_1)+\varphi_2(x_2,x_3,s_2))} a_1(x_1, x_2, s_1, \hbar) \right.$$
$$\left. \times a_2(x_2, x_3, s_2, \hbar) f(x_1)\, ds_1 dx_1 ds_2 dx_2 \right) dx_3^{\frac{1}{2}}.$$

Since $\varphi_1(x_1, x_2, s_1) + \varphi_2(x_2, x_3, s_2)$ is a normalized generating function of $\Gamma_2 \circ \Gamma_1$ with respect to the fibration

$$X_1 \times X_3 \times (X_2 \times S_1 \times S_2) \to X_1 \times X_2$$

with fiber dimension $d_1 + d_2 + n_2$, the conclusion follows. $\square$

As a consequence, we see if $P_i \in \Psi^{m_i}(X)$, then $P_1 P_2 \in \Psi^{m_1+m_2}(X)$, a fact that we have used many times in this course.

5. The symbol calculus

¶ The space of symbol: a local description.

Let $\mu = u\nu \in I^k(X, \Lambda)$, where ν is a nowhere vanishing half density on X. We can regard μ as a semiclassical Fourier integral operator from pt to X. Locally

$$u(x, \hbar) = \hbar^{k-\frac{d}{2}} \int a(x, s, \hbar) e^{\frac{i}{\hbar}\varphi(x,s)} ds,$$

where $\varphi = \varphi(x, s)$ is a generating function with respect to a fibration $X \times S \to X$, and $a \in C_0^\infty(X \times S \times \hbar)$. Recall that the critical set C_φ is the subset of $X \times S$ defined by the equations

$$\frac{\partial \varphi}{\partial s_i} = 0, \quad i = 1, 2, \cdots, d.$$

Proposition 5.1. *If $a(x, s, 0) = 0$ on C_φ, then $\mu \in I_0^{k+1}(X, \Lambda)$.*

Proof. If $a(x, s, 0) = 0$ on C_φ, then we can write

$$a(x, s, \hbar) = \sum a_j(x, s, \hbar) \frac{\partial \varphi}{\partial s_j} + a_0(x, s, \hbar)\hbar \tag{16}$$

for smooth functions $a_0, \cdots, a_d$. So we can rewrite the integral expression of $u(x, \hbar)$ as $u_0 + u_1 + \cdots + u_d$, where

$$u_0(x, \hbar) = \hbar^{k+1-\frac{d}{2}} \int a_0(x, s, \hbar) e^{\frac{i}{\hbar}\varphi(x,s)} ds,$$

and

$$\begin{aligned} u_j &= \hbar^{k-\frac{d}{2}} \int a_j(x, s, \hbar) \frac{\partial \varphi}{\partial s_j} e^{\frac{i}{\hbar}\varphi} ds \\ &= -i\hbar^{k+1-\frac{d}{2}} \int a_j(x, s, \hbar) \frac{\partial}{\partial s_j} e^{\frac{i}{\hbar}\varphi} ds \\ &= i\hbar^{k+1-\frac{d}{2}} \int (\frac{\partial}{\partial s_j} a_j(x, s, \hbar)) e^{\frac{i}{\hbar}\varphi} ds. \end{aligned}$$

□

Now suppose we have all the data as before. Recall that p_φ is a diffeomorphism from C_φ onto Λ. For any $\mu \in I^k(X, \Lambda)$ we tentatively define its "symbol" to be the function $\sigma_\varphi(\mu) \in C^\infty(\Lambda)$ such that

$$\sigma_\varphi(\mu)(x, \xi) = a(x, s, 0), \quad \text{if } (x, s) \in C_\varphi \text{ and } p_\varphi(x, s) = (x, \xi). \tag{17}$$

Of course the function $\sigma_\varphi(\mu)$ depends on the choice of fibration, the choice of generating function, and also on the choice of the non-vanishing half density on X.

Proposition 5.2. *For any $p \in \Lambda$ the assertion $\sigma_\varphi(\mu)(p) = 0$ is intrinsic, i.e. independent of all the choices above.*

Proof. We have three cases to check:

- Changing the choice of non-vanishing half densities $\rightsquigarrow$ multiplying $\sigma_\varphi(\mu)(p)$ by a nonzero constant.
- Hormander move II (equivalence) will not change the symbol.
- Hormander move III (adding variable) $\rightsquigarrow$ use the lemma of stationary phase to get a nonzero coefficient.

□

¶ The intrinsic symbol.

It follows that

$$I^k_p(X,\Lambda) = \{\mu \in I^k(X,\Lambda) \mid \sigma_\varphi(\mu)(p) = 0\} \tag{18}$$

is intrinsically defined. We thus get a line bundle $\mathbb{L} \to \Lambda$, whose fiber over p is

$$\mathbb{L}_p = I^k(X,\Lambda)/I^k_p(X,\Lambda). \tag{19}$$

Note that the fiber $\mathbb{L}_p$ is independent of k, since multiplication by $\hbar^{l-k}$ is an isomorphism from $I^k(X,\Lambda)$ to $I^l(X,\Lambda)$ which maps $I^k_p(X,\Lambda)$ to $I^l_p(X,\Lambda)$. Moreover, a choice of data above gives a trivialization of $\mathbb{L} \to \Lambda$. So it is a smooth line bundle.

Now we can define the intrinsic symbol of an oscillatory half density:

Definition 5.3. The *intrinsic symbol* of $\mu \in I^k(X,\Lambda)$ is a section $\sigma(\mu)$ of the line bundle $\mathbb{L}$ given by

$$\sigma(\mu)(p) = [\mu_p] \in \mathbb{L}_p. \tag{20}$$

Obviously if $\mu \in I^k(X,\Lambda)$ and $\sigma(\mu) = 0$, then $\mu \in I^{k+1}(X,\Lambda)$.

For a semi-classical FIO $F = F_\mu \in \mathcal{F}^m_0(\Gamma)$. We can define a line bundle $\mathbb{L}_\Gamma \to \Gamma$ to be the pull back of the line bundle $\mathbb{L}_\Lambda \to \Lambda$ via the map $id \times \sigma_2$. We then define the symbol of F to be the section of this new line bundle

$$\sigma(F) = (id \times \sigma_2)^* \sigma(\mu).$$

Remark. We can calculate the symbol of the composition of two transversally composable semiclassical Fourier integral operators. More precisely, if we let $\Gamma_1 \in T^*X_1 \times T^*X_2^-$ and $\Gamma_2 \in T^*X_2 \times T^*X_3^-$ be exact canonical relations that are transversally composable and let $\Gamma = \Gamma_2 \circ \Gamma_1$. Then the map

$$\alpha : F \to \Gamma, \quad (m_1, m_2, m_3) \to (m_1, m_3)$$

is a diffeomorphism. Let

$$j : F \to \Gamma_1 \times \Gamma_2, (m_1, m_2, m_3) \mapsto ((m_1, m_2), (m_2, m_3))$$

be the canonical embedding. Then

Theorem 5.4. *If $F_i \in \mathcal{F}^{m_i}(\Gamma_i)$, then*

$$\alpha^*(\sigma(F_2 \circ F_1)) = j^*(\sigma(F_1)\sigma(F_2)). \tag{21}$$

For a proof, c.f. Guillemin-Sternberg §8.4.

¶ Semi-classical pseudo-differential operators revisited.

Now we apply the symbol theory we just described to semi-classical pseudo-differential operators on manifolds. Recall that the space of semi-classical pseudo-differential operators is by definition $\Psi^m(X) = \mathcal{F}^m(\Delta_M)$, where $\Delta_M = \Gamma_{id}$ is the diagonal (which is a Lagrangian) in $M \times M^-$. Note that we have

$$\Delta_M \circ \Delta_M = \Delta_M.$$

It follows that if $P_i \in \Psi^{m_i}(X)$, then $P_2 \circ P_1 \in \Psi^{(m_1+m_2)}(X)$.

We can identify M with Δ_M and also with the set $F = \{(m,m,m)\}$ under the obvious maps. Then the maps $\alpha : F \to \Delta_M \circ \Delta_M = \Delta_M$ and $pr_i : F \to \Delta_M$ are all identities. So according to the composition law, we have a canonical isomorphism

$$\mathbb{L}_M \simeq \mathbb{L}_M \otimes \mathbb{L}_M, \tag{22}$$

where $\mathbb{L}_M$ is the pull back of $\mathbb{L}_{\Delta_M}$ to M under the diagonal embedding. By tenser both sides with $\mathbb{L}_M^{-1}$, we see $\mathbb{L}_M$ is a trivial line bundle, and thus the sections of $\mathbb{L}_M$ can be identified with $C^\infty(M)$. Under this identification, the symbol map is a map

$$\sigma_m : \Psi^m(X) \to C^\infty(M)$$

and the kernel of this symbol map is $\Psi^{m+1}(X)$, and the composition law becomes

$$\sigma_{m_1+m_2}(P_2 \circ P_1) = \sigma_{m_2}(P_2)\sigma_{m_1}(P_1).$$

¶ An application: Egorov's theorem.

Now let $\gamma : T^*X_1 \to T^*X_2$ be a symplectomorphism, and set

$$\Gamma_1 = \text{graph}(\gamma), \quad \Gamma_2 = \text{graph}(\gamma^{-1}).$$

Suppose F_1 is an invertible semi-classical Fourier integral operator associated to Γ_1 whose inverse $F_2 = F_1^{-1}$ is a semi-classical Fourier integral operator associated to Γ_2.

Theorem 5.5 (Egorov Theorem)**.** *For any $A \in \Psi^k(X_2)$, we have*

$$F_2 \circ A \circ F_1 \in \Psi^k(X_1) \tag{23}$$

and

$$\sigma(F_2 \circ A \circ F_1) = \gamma^*(\sigma(A)). \tag{24}$$

Proof. The first assertion follows from the fact (check)

$$\Gamma_2 \circ \Delta_{T^*X_2} \circ \Gamma_1 = \Delta_{T^*X_1}.$$

For the second one, if we denote $\alpha : F \simeq \{(m_1, m_2) \mid m_2 = \gamma(m_1), m_1 \in M_1\} \to M_1$ be the standard projection, and $j : F \to \Gamma_2 \times \Delta_{M_2} \times \Gamma_1$ be the map that sends $(m_1, m_2 = \gamma(m_1))$ to $(m_2, m_1, m_1, m_1, m_1, m_2)$, then the expression $\alpha^*(\sigma(F_2 \circ F_1)) = j^*(\sigma(F_1)\sigma(F_2))$ becomes

$$\sigma(F_2 A F_1)(x,\xi) = \sigma(F_2)(y,\eta,x,\xi)\sigma(A)(y,\eta)\sigma(F_1)(x,\xi,y,\eta)$$

for $(x, \xi, y, \eta) \in \Gamma_1$, i.e. for $(y, \eta) = \gamma(x, \xi)$. Since $\sigma(A)$ can be identified with a scalar, we can pull the middle term out of the product. On the other hand, since $F_2 \circ F_1 = 1$, we have

$$\sigma(F_2)(y, \eta, x, \xi)\sigma(F_1)(x, \xi, y, \eta) = 1.$$

It follows

$$\sigma(F_2 A F_1)(x, \xi) = \sigma(A)(y, \eta),$$

where $(y, \eta) = \gamma(x, \xi)$. □

¶The end which is not the end.

Unfortunately we don't have time to fill in all the proofs of the basic theory of semi-classical Fourier integral operators in the previous five lectures, nor do we have time to touch any other interesting topics – L^p-theory, trace formulae, propagation of singularities etc. As one can imagine, semiclassical Fourier integral operators, as a substantial generalization of pseudodifferential operators, has wide applications to PDEs. Let's end this brief introduction to FIOs by mentioning a number of interesting results in spectral geometry which can be proved with the help of FIOs:

- (Weyl law) In Lecture 23 we showed that on compact manifolds, for a pseudodifferential operator P satisfying suitable conditions mentioned, one has

  $$\#(\mathrm{Spec}(P) \cap [a, b]) = \frac{1}{(2\pi\hbar)^n}\left(\mathrm{Vol}(p^{-1}([a, b])) + o(1)\right).$$

  As one of the first applications of the theory of Fourier integral operators, one can prove [3]

  Theorem 5.6 (Sharp Weyl law)**.** *Under the assumptions of Weyl law, we have*

  $$\#(\mathrm{Spec}(P) \cap [a, b]) = \frac{1}{(2\pi\hbar)^n}\left(\mathrm{Vol}(p^{-1}([a, b])) + O(\hbar)\right).$$

  According to a theorem of Duistermaat-Guillemin, under suitable assumptions (the set of closed trajectories of the Hamiltonian flow has measure zero), the error term $O(\hbar)$ can be improved to $o(\hbar)$. (So in general the error term depends heavily on the dynamical behavior of the Hamiltonian flow.)
- Instead of counting eigenvalues in an interval $[a, b]$ of fixed length, one may also counts eigenvalues in a "shrinking" interval $[E - c\hbar, E + c\hbar]$:

  Theorem 5.7 (Local Weyl law)**.** *Assume that the set of periodic points of the Hamiltonian flow of p has Liouville measure zero in $\Sigma = p^{-1}(E)$. Denote*

  $$J_\hbar = \{j \mid \lambda_{j,\hbar} \in [E - c\hbar, E + c\hbar]\}.$$

[3] The theorem was first proved by L. Hörmander in the microlocal setting.

Then $\#J_\hbar \sim C\hbar^{1-n}\mathrm{Vol}(\Sigma)$, *and*

$$\frac{1}{\#J_\hbar}\sum_{j\in J_\hbar}\langle A\varphi_j,\varphi_j\rangle \to \frac{1}{\mathrm{Vol}(\Sigma)}\int_\Sigma \sigma(A)d\mu_E.$$

(Compare: The Generalized Weyl law in Lecture 23)

As an application, one can prove the quantum ergodicity on one level set Σ:

Theorem 5.8 (Quantum ergodicity)**.** *Assume that the set of periodic points of the Hamiltonian flow of* p *has Liouville measure zero in* $\Sigma = p^{-1}(E)$*. If the flow is ergodic on* Σ*, then the quantum ergodicity theorem holds:*

$$\lim_{\hbar\to 0}\frac{1}{\#J_\hbar}\sum_{j\in J_\hbar}\left(\langle A\varphi_j,\varphi_j\rangle - \frac{1}{\mathrm{Vol}(\Sigma)}\int_\Sigma \sigma(A)d\mu_E\right)^2 = 0.$$

(Similarly one can write down a density-one subsequence version.)

I hope I have explained clearly enough basic ideas in semiclassical analysis. It absorb nutrient from the classical-quantum correspondence in modern physics. It builds a bridge between analysis and geometry. It is still a very active research area of mathematics that need to be explored.

Now this is not the end. It is not even the beginning of the end. But it is, perhaps, the end of the beginning. **Winston Churchill**