PROBLEM SET 1 SEMICLASSICAL MICROLOCAL ANALYSIS
DUE: OCT. 19, 2020

(1) [Poisson bracket]
Prove that the Poisson bracket $\{\cdot,\cdot\}$ turns $C^\infty(T^*\mathbb{R}^n)$ into *Poisson algebra*, namely for any $f, g, h \in C^\infty(T^*\mathbb{R}^n)$,
 (a) (Anti-commutativity) $\{f, g\} = -\{g, f\}$.
 (b) (Jacobi's identity) $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.
 (c) (Leibniz's rule) $\{f, gh\} = \{f, g\}h + g\{f, h\}$.
Moreover, prove that
 (d) (Commutator relation) $\Xi_{\{f,g\}} = [\Xi_f, \Xi_g]$.
[Ref: Zworski, *Semiclassical Analysis*, Lemma 2.9]

(2) [Ehrenfest Theorem]
Prove the Ehrenfest Theorem that relates the time derivative of the expectation values of the position and momentum operators to the expectation value of the force $-\nabla V$:

$$\begin{cases} \frac{d}{dt}\langle Q_j\rangle_{\psi(t)} = \langle P_j\rangle_{\psi(t)}, \\ \frac{d}{dt}\langle P_j\rangle_{\psi(t)} = -\langle \partial_j V\rangle_{\psi(t)}. \end{cases}$$

[Ref: Hall, *Quantum theory for mathematicians*, Section 3.7.5]

(3) [No-go theorem]
Complete the proof of the no-go theorem as outlined in Lecture 3. Namely, prove that one can't quantize all polynomials in x and ξ of degree ≤ 4 if we assume Dirac's axioms (D1)-(D4) listed in the Lecture 3. For this purpose we let $Q(x)$ and $Q(\xi)$ be the quantizations of the position function x and the momentum function ξ respectively.
 (a) Prove: There exists constant c such that

$$Q(x\xi) = Q(x)Q(\xi) + c \cdot \mathrm{Id}.$$

 (b) Inductively prove that for any $m \in \mathbb{N}$, we have

$$Q(x^m) = Q(x)^m \quad \text{and} \quad Q(\xi^m) = Q(\xi)^m.$$

 (c) Compute $\{\xi^2, x^3\}$ and $\{x^2, \xi^3\}$, and prove

$$Q(x^2\xi) = \frac{Q(x)^2Q(\xi) + Q(\xi)Q(x)^2}{2}, \quad Q(x\xi^2) = \frac{Q(x)Q(\xi)^2 + Q(\xi)^2Q(x)}{2}$$

 (d) Compute $\{x^2\xi, x\xi^2\}$ and $\{x^3, \xi^3\}$, and deduce a contradiction.
[Ref: My 2014 notes, Lecture 1.]

(4) [Hermite polynomials]
Consider the Harmonic oscillator with $n = 1$. Denote $u_j(x) = C^j u_0$ be the eigenfunction of $\hat{H}$ associated with the eigenvalue $(j + \frac{1}{2})\hbar$, where $C = \frac{1}{\sqrt{2}}\left(\frac{\hbar}{\sqrt{-1}}\frac{\partial}{\partial x} + \sqrt{-1}x\right)$ is the creation operator, and $u_0(x) = e^{-x^2/2\hbar}$. To get an explicit expression, write

$$u_j(x) = \left(\frac{\sqrt{-1}}{\sqrt{2}}\right)^j \hbar^{\frac{j}{2}} H_j\left(\frac{x}{\sqrt{\hbar}}\right) e^{-\frac{x^2}{2\hbar}}.$$

(a) Calculate H_0, H_1, H_2 and H_3.
(b) Use the facts $Cu_j = u_{j+1}$ to prove $H_{j+1}(x) = 2xH_j(x) - H_j'(x)$. As a consequence, H_j is a polynomial of degree j, called *Hermite polynomials*.
(c) First prove $Au_j = ju_{j-1}$, then prove $H_j'(x) = 2jH_{j-1}(x)$. As a consequence, Hermite polynomials satisfy the recurrence relation $H_{j+1} = 2xH_j(x) - 2jH_{j-1}(x)$.
(d) Prove the Rodrigues' formula for H_n:

$$H_j(x) = (-1)^j e^{x^2} \left(\frac{d}{dx}\right)^j (e^{-x^2}).$$

(e) Prove that H_n has the following exponential generating function

$$e^{2xt-t^2} = \sum_{j=0}^{\infty} H_j(x) \frac{t^j}{j!}.$$

[Ref: Hall, *Quantum theory for mathematicians*, Chapter 11.]

(5) [The uncertainty principle]
Let A be a quantum observable, ψ a quantum state, and $a \in \mathbb{R}$. The expression

$$\delta_a^\psi(A) := \sqrt{\langle (A - a \cdot \mathrm{Id})^2 \rangle_\psi}$$

can be used as a measure of how much the observable A in the state ψ fails to be concentrated at a. In particular, if we take $a = \langle A \rangle_\psi$, then we write $\delta^\psi(A) := \delta^\psi_{\langle A \rangle_\psi}(A)$ (which is just the standard deviation) and call it the *uncertainty* of A in a state ψ.
(a) Prove: For any self-adjoint operators A and B, any $\psi \in \mathrm{Dom}(AB) \cap \mathrm{Dom}(BA)$ and any $a, b \in \mathbb{R}$, we have

$$\delta_a^\psi(A)\delta_b^\psi(B) \geq \frac{1}{2} \left| \langle [A, B] \rangle_\psi \right|.$$

(b) Let $Q = x$ and $P = \frac{\hbar}{i}\frac{d}{dx}$ be the canonical quantum observables (acting on $L^2(\mathbb{R})$) associated with the position x and momentum ξ. Prove: For $\psi \in \mathrm{Dom}(QP) \cap \mathrm{Dom}(PQ)$,

$$\delta_a^\psi(Q)\delta_b^\psi(P) \geq \frac{\hbar}{2}.$$

[Ref: Folland, *Harmonic analysis in phase space*, Theorem 1.34]
[Ref: Hall, *Quantum theory for mathematicians*, Chapter 12.]

(6) [Semiclassical Fourier transform]
Define the semiclassical Fourier transform of a function $\varphi \in \mathscr{S}$ to be

$$\mathcal{F}_\hbar\varphi(\xi) := (\mathcal{F}\varphi)(\frac{\xi}{\hbar}) = \int_{\mathbb{R}^n} e^{-\frac{ix\cdot\xi}{\hbar}}\varphi(x)dx.$$

(a) Write down the analogue statements of Proposition 1.2 of Lecture 4 for $\mathcal{F}_\hbar$, and prove them directly (namely not as a consequence of Proposition 1.2).
(b) Write down the semiclassical versions of Theorem 1.3, Corollary 1.4 and Theorem 2.3 (for which we compute the semiclassical Fourier transform of $e^{\frac{i}{2\hbar}x^TQx^T}$).
(c) Let $P_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j}$ be the momentum operator. Prove: For $\varphi \in \mathscr{S}$,

$$\langle P_j\rangle_\varphi = \int_{\mathbb{R}^n} \xi_j|\mathcal{F}_\hbar\varphi(\xi)|^2 d\xi.$$

(d) Prove the following semiclassical version of uncertainty principle: For any $\varphi \in \mathscr{S}$,

$$\|x_j\varphi\|_{L^2} \cdot \|\xi_j\mathcal{F}_\hbar\varphi\|_{L^2} \geq \frac{\hbar}{2}\|\varphi\|_{L^2} \cdot \|\mathcal{F}_\hbar\varphi\|_{L^2}.$$

(e) Figure out the the condition for the equality in (d).
[Ref: Zworski, *Semiclassical Analysis*, §3.3]

(7) [Poisson summation formula]
Let $f \in \mathscr{S}(\mathbb{R}^n)$. Define $h(x) := \sum_{v\in\mathbb{Z}^n} f(x+v)$.
(a) Prove: the function h is smooth and periodic with period $\mathbb{Z}^n$.
(b) Expand h into its Fourier series, and prove:

$$\sum_{v\in\mathbb{Z}^n} f(v) = \sum_{\mu\in\mathbb{Z}^n} \hat{f}(2\pi\mu).$$

[Ref: Guillemin-Sternberg, *Semi-classical Analysis*, §15.10.4.]
[As an application, read §15.10.1, §15.10.3 and §15.11.]

(8) [Almost analytic extension]
For any $f \in \mathscr{S}(\mathbb{R})$, we say a function $\tilde{f} \in C^\infty(\mathbb{C})$ is an *almost analytic extension* of f if it is an extension of f which supports near the real axis such that $\bar{\partial}_z\tilde{f}$ vanishes to infinite order near the real axis, namely

$$\tilde{f}|_{\mathbb{R}} = f, \quad \mathrm{supp}\tilde{f} \subset \{z : |\mathrm{Im}(z)| \leq 1\} \quad \text{and} \quad \bar{\partial}_z\tilde{f}(z) = O(|\mathrm{Im}(z)|^\infty).$$

(a) Fix any cut-off function $\chi \in C_0^\infty((-1,1))$ with $\chi \equiv 1$ on $[-1/2, 1/2]$. Prove:

$$\tilde{f}(z) := \frac{1}{2\pi}\chi(y)\int_{\mathbb{R}} \chi(y\xi)\hat{f}(\xi)e^{i\xi(x+iy)}d\xi$$

is an almost analytic extension of f.
(b) Let $\tilde{f}$ be an almost analytic extension of f. Use Green's formula to prove:

$$f(t) = \frac{1}{\pi i}\int_{\mathbb{C}} \frac{\bar{\partial}_z\tilde{f}(z)}{t-z}dz.$$

[Ref: Zworski, *Semiclassical Analysis*, Theorem 3.6 and Theorem 14.8.]