PROBLEM SET 2 SEMICLASSICAL MICROLOCAL ANALYSIS
DUE: NOV. 16, 2020

(1) [The Wigner Transform]
Given $f, g \in \mathscr{S}(\mathbb{R}^n)$, define the Fourier-Wigner transform of f, g to be the function

$$V(f,g)(p,q) := \frac{1}{(2\pi\hbar)^n} \langle e^{\frac{i}{\hbar}(q\cdot Q + p \cdot P)} f, g\rangle$$

and define their Wigner transform to be the function

$$W(f,g)(x,\xi) = (\mathcal{F}_\hbar)_{(p,q)\to(x,\xi)} V(f,g).$$

(a) Prove:

$$V(f,g)(p,q) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} y\cdot q} f(y+\frac{p}{2}) \overline{g(y-\frac{p}{2})}\ dy.$$

(b) Prove:

$$W(f,g)(x,\xi) = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} p \cdot \xi} f(x+\frac{p}{2}) \overline{g(x-\frac{p}{2})}\ dp.$$

(c) Prove: For any $a \in \mathscr{S}$,

$$\langle \widehat{a}^W f, g\rangle = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a(x,\xi) W(f,g)(x,\xi)\ dx d\xi.$$

(d) Prove: [Moyal's identity]

$$\langle W(f_1,g_1), W(f_2,g_2)\rangle = (2\pi\hbar)^n \langle f_1, f_2\rangle \overline{\langle g_1, g_2\rangle}.$$

(e) Prove:

$$[\widehat{W(f,g)}^W \varphi](x) = \langle \varphi, g\rangle f(x).$$

In particular, $\widehat{W(f,f)}^W \varphi$ is the projection of φ onto f.

[Ref: Folland, *Harmonic analysis on phase space*]

(2) [Symplectic invariance of the Weyl quantization]
We mentioned three special cases of the symplectic invariance of the Weyl quantization, proved the case (A) and a special case of (B) (with C=Id). (Lec 7, p. 4)

(a) Prove case (B) for general symmetric matrix C.
(b) Prove case (C).
(c) Prove: for any $a, b \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we have

$$[(a\cdot x + \widehat{b\cdot \xi} + c)^m]^W = (a \cdot Q + b\cdot P + c\ \mathrm{Id})^m.$$

(Hint: First prove the identity with $c = 0$.)

[Ref: Folland, *Harmonic analysis on phase space*.]

(3) [An counterexample to uncertainty and BCH]
Consider the Hilbert space $\mathcal{H} = L^2([-1,1])$. Note that

$$\psi_n(x) := \frac{1}{\sqrt{2}} e^{inx}, \quad n \in \mathbb{Z}$$

form an orthonormal basis of $\mathcal{H}$. Define

$$Q : \mathcal{H} \to \mathcal{H}, \quad \varphi \mapsto x\varphi$$

and

$$P : \mathrm{Dom}(P) \subset \mathcal{H} \to \mathcal{H}, \quad \varphi \mapsto \frac{\hbar}{i}\frac{d\varphi}{dx},$$

where $\mathrm{Dom}(P) = \{\varphi \in C^1([-1,1]) \mid \varphi(1) = \varphi(-1)\}$.

(a) Prove: Q is self-adjoint on $\mathcal{H}$, and P is essentially self-adjoint on $\mathrm{Dom}(P)$.
(b) Prove: The uncertainty (c.f. PSet1-5) $\delta^{\psi_n}(Q)\delta^{\psi_n}(P) = 0$. Explain why this does not violate PSet1-5-(b).
(c) For any $a \in \mathbb{R}$, define $S_a : \mathcal{H} \to \mathcal{H}$ by

$$(S_a\varphi)(x) := \varphi(x + a - 2m_{x,a}),$$

where $m_{x,a}$ is the unique integer such that $x + a - 2m_{x,a} \in [-1, 1)$. Prove: for any $t \in \mathbb{R}$, $e^{itB/\hbar} = S_t$.
(d) Prove: Q, P violate the Baker-Campbell-Hausdorff formula. In other words, $[Q,[Q,P]] = [P,[Q,P]] = 0$ but $e^{\frac{it(Q+P)}{\hbar}} \neq e^{-[\frac{itQ}{\hbar}, \frac{itP}{\hbar}]} e^{\frac{itQ}{\hbar}} e^{\frac{itP}{\hbar}}$.

[Ref: Hall, *Quantum theory for mathematicians*, §12.2 and Example 14.5.]

(4) [Mehler formula]
For a symbol $a(x,\xi)$, although we proved the following equality for several special cases (e.g. a is linear or $a(x,\xi) = ip(\xi)$, where p is a real-valued polynomial), in general (where we assume $e^{ta(x,\xi)}$ is also a nice symbol function)

$$e^{t\widehat{a}^W} \neq \widehat{e^{ta}}^W.$$

In this problem let's explore the case $a(x,\xi) = -H(x,\xi) = -\frac{x^2+\xi^2}{2}$, where for simplicity we assumed $n = 1$. Then $e^{t\widehat{a}^W} = e^{-t\widehat{H}}$, where $\widehat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + \frac{x^2}{2}$ be the 1-dim harmonic oscillator. We want to find a function $q_t(x,\xi,t)$ so that

$$e^{-t\widehat{H}} = \widehat{e^{q_t}}^W.$$

(a) Prove: $\frac{\partial q_t}{\partial t} e^{q_t} = -H \star e^{q_t}$.
(b) For the non-semiclassical setting (namely, there is no $\hbar$), I saw from literature a formula

$$e^{q_t} = \frac{1}{\cosh\frac{t}{2}} e^{-(x^2+\xi^2)\tanh\frac{t}{2}}.$$

Please write down a semiclassical version of this formula, and check it via (a).
(c) Write down the Schwartz kernel of the operator $e^{-t\widehat{H}}$.

(5) [t-quantizations]
Prove the following properties of semiclassical t-quantizations:
 (a) Prove the formulas (10) and (11) in Lecture 9, page 9.
 (b) Prove Theorem 3.2 in Lecture 9.
 (c) Prove: For $a \in \mathscr{S}(\mathbb{R}^{2n})$, if we let $b(x,\xi) = e^{i\hbar D_x \cdot D_\xi}\overline{a}(x,\xi)$, then $(\widehat{a}^W)^* = \widehat{b}^W$.
[Ref: Zworski, *Semiclassical Analysis*, §4.3.3 and §4.3.4]

(6) [Exact quantization condition]
Prove Proposition 2.2 and Proposition 2.3 in Lecture 9 (page 7).
Hint: Use the commutative relation $[A, BC] = [A, B]C + B[A, C]$ for operators.
[Ref: Hall, *Quantum theory for mathematicians*, Proposition 13.11.]

(7) [Oscillatory testing for standard quantization]
Suppose m is an order function, and $a \in S_\delta(m)$. Prove:

$$a(x,\xi) = e^{-\frac{i}{\hbar}x\cdot\xi}\ \widehat{a}^{KN}(e^{\frac{i}{\hbar}x\cdot\xi}).$$

So one can easily recover the symbol $a(x,\xi)$ from its Kohn-Nirenberg quantization!
[Ref: Zworski, *Semiclassical Analysis*, Theorem 4.19]

(8) [Semiclassical differential operators]
A semiclassical differential operator is an operator of the form

$$P = \sum_{i=0}^{m} \hbar^i \sum_{|\alpha|\le k_i} a_{i,\alpha}(x)(\hbar D)^\alpha,$$

where m, k_i's are positive integers. Prove: A semiclassical pseudodifferential operator is a semiclassical differential operator if and only if it is the Weyl quantization of a function which is a polynomial in both ξ and $\hbar$.
[Ref: Folland, *Harmonic analysis on phase space*, Prop. 2.11.]

(9) [Dirichlet-to-Neumann map as pseudodifferential operator]
Consider the upper half space $\mathbb{R}^{n+1}_+ := \mathbb{R}_+ \times \mathbb{R}^n$. We use t as the coordinate on $\mathbb{R}_+$ and $x = (x_1, \cdots, x_n)$ as the coordinates on $\mathbb{R}^n$. The Dirichlet-to-Neumann operator $\Lambda : \mathscr{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is defined as follows: Given any $f \in \mathscr{S}(\mathbb{R}^n) = \mathscr{S}(\partial\mathbb{R}^{n+1}_+)$, let $u = u(t,x) \in C^\infty(\mathbb{R}^{n+1}_+)$ be the solution to the equation

$$\begin{cases} (\hbar^2\partial_t^2 + \hbar^2\Delta)u(t,x) = 0, \\ u(0,x) = f(x). \end{cases}$$

which decays rapidly as $t \to +\infty$. Define $\Lambda(f)$ to be the exterior normal derivative

$$\Lambda(f) = -\hbar \left.\frac{\partial u}{\partial t}\right|_{t=0}.$$

It turns out that Λ is a semiclassical pseudodifferential operator with symbol $|\xi|$:
 (a) Prove: $\hat{u}(t,\xi) = Ce^{-t|\xi|/\hbar}\hat{f}(\xi)$, where $\hat{u}(t,\xi) = [(\mathcal{F}_\hbar)_{x\to\xi}u](t,\xi)$.
 (b) Prove: $\Lambda = \mathcal{F}_\hbar^{-1} \circ |\xi| \circ \mathcal{F}_\hbar$.