PROBLEM SET 3 SEMICLASSICAL MICROLOCAL ANALYSIS
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(1) [Schur's test for $L^p(\mathbb{R}^n)$]
We can extend Schur's test to prove the boundedness of an integral operator on the space $L^p(\mathbb{R}^n)$. More precisely, let $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a continuous function satisfying

$$C_1 = \sup_x \int_{\mathbb{R}^n} |K(x,y)| dy < +\infty \qquad \text{and} \qquad C_2 = \sup_y \int_{\mathbb{R}^n} |K(x,y)| dx < +\infty,$$

Let A_K be the integral operator with Schwartz kernel K. Then for any $1 \le p \le +\infty$,

$$\|A_K f\|_{L^p(\mathbb{R}^n)} \le C_1^{1/p} C_2^{1/p'} \|f\|_{L^p(\mathbb{R}^n)},$$

where as usual, $\frac{1}{p} + \frac{1}{p'} = 1$.
[Ref: T. Tao, *Interpolation, Schur's test, Young's inequality, Hausdorff-Young, Chris-Kiselev*, https://www.math.ucla.edu/∼tao/247a.1.06f/notes2.pdf, in §5 three different proofs are given.]

(2) [Pseudolocality]
Suppose $a \in S(1)$, $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^n)$ such that $\mathrm{supp}\chi_0 \cap \mathrm{supp}\chi_1 = \emptyset$. Prove:

$$\|\chi_1 \widehat{a}^W \chi_2\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = O(\hbar^\infty).$$

(3) [Application of Beals theorem: Solving an operator equation]
Fix $\hbar = 1$ in this problem. Suppose $c_t \in S(1)$ is a family of symbols which is continuous in t for $|t| \le t_0$. For any $q_0 \in S(1)$, consider the operator equation

$$\begin{cases} (\partial_t + \widehat{c(t)}^W) Q(t) = 0, \\ \qquad\qquad Q(0) = \widehat{q_0}^W \end{cases}.$$

According to some advanced version of Picard theorem in ODE (in suitable Banach space), there exists a unique solution $Q(t) \in \mathcal{L}(L^2(\mathbb{R}^n))$ for $|t| \le t_0$. Prove: The solution $Q(t)$ has the form $Q(t) = \widehat{q_t}^W$ with $q_t \in S(1)$.
[Ref: Zworski, *Semiclassical Analysis*. Lemma 8.5]

(4) [Square root of positive $\hbar$-PsDO]
Suppose $a \in S(m)$ is elliptic and positive-valued. Prove: There exists $b \in S(\sqrt{m})$ such that

$$b \star b = a + O(\hbar^\infty).$$

As a consequence, $(\widehat{b}^W)^2 = \widehat{a}^W + O(\hbar^\infty)$.
Hint: Use the method in the proof of Theorem 1.3 in Lecture 14.

(5) [Trace class $\hbar$-PsDOs]
Prove Proposition 2.6 in Lecture 13.
[Ref: Dimassi-Sjostrand, *Spectral Asymptotics in the Semi-Classical Limit*, §9]

(6) [Anti-Wick quantization]
Let $\Phi_0(x) = (\pi\hbar)^{-n/4}e^{x^2/2\hbar}$. Note that $\|\Phi_0\|_{L^2} = 1$. Let $P_0 : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the orthogonal projection on the vector Φ_0, namely

$$\Phi_0(u) = \langle u, \Phi_0 \rangle \Phi_0.$$

(a) Prove P_0 is a semiclassical pseudodifferential operator and find its Weyl symbol.
(b) For any $(x_0, \xi_0) \in \mathbb{R}^n$, define a unitary operator $U_{x_0,\xi_0} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$U_{x_0,\xi_0}u(x) = e^{ix\cdot\xi_0}u(x - x_0).$$

Define $P_{x_0,\xi_0} := U_{x_0,\xi_0}P_0U^*_{x_0,\xi_0}$. Prove: P_{x_0,ξ_0} is a positive operator, i.e.

$$\langle P_{x_0,\xi_0}u, u\rangle \geq 0, \quad \forall u \in \mathscr{S}.$$

(c) For any $a \in S(m)$, define the anti-Wick quantization of a to the the operator

$$\widehat{a}^{\text{anti-Wick}}u(y) := \int_{\mathbb{R}^{2n}} a(x,\xi)P_{x,\xi}u(y)dxd\xi,$$

Prove: the anti-Wick quantization is a positive quantization, i.e. if $a \geq 0$, then $\widehat{a}^{\text{anti-Wick}}$ is a positive operator.
(d) Prove: The $\widehat{a}^{\text{anti-Wick}}$ is a semiclassical pseudodifferential operator whose Weyl symbol is

$$\tilde{a}(x,\xi) = (\pi\hbar)^{-n}\int_{\mathbb{R}^{2n}} a(y,\eta)e^{-[(x-y)^2+(\xi-\eta)^2]/\hbar}dyd\eta.$$

(e) Prove: If $a \in S(1)$, then $\tilde{a} \in S(1)$ and $\tilde{a} - a \in \hbar S(1)$.
(f) Prove: For $a \in S(1)$, $\|\widehat{a}^{\text{anti-Wick}} - \widehat{a}^W\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = O(\hbar)$.
(g) Prove the Sharp Garding inequality (Theorem 2.1 in Lecture 15) via the anti-Wick quantization.

[Ref: Shubin, Pseudodifferential operators and spectral theory, §24]
[Ref: Wong, *An introduction to semiclassical and microlocal analysis*, Problem 22 on page 67]