**PROBLEM SET 4** **SEMICLASSICAL MICROLOCAL ANALYSIS**
**DUE: DEC. 28, 2020**

(1) [Adjoint in KN quantization]
  (a) Suppose $m$ is an order function, $a \in S(m)$. Find the Kohn-Nirenberg symbol of the adjoint $(\widehat{a}^{KN})^*$.
  (b) Suppose $k$ is a real number and $a \in S^k$ is an invariant symbol. Show that $(\widehat{a}^{KN})^*$ is also in $\Psi^k$.

(2) [PsDO on manifolds]
Prove Proposition 2.2 in Lecture 20.

(3) [Dyadic P.O.U.]
Prove Lemma 2.6 in Lecture 18.
[ Ref: Zworski, *Semiclassical Analysis*. Lemma 7.14]

(4) [Elliptic Estimate]
  (a) Suppose $a \in S(1)$ and $p \in S(m)$, where $m \geq 1$ is an order function. Moreover, suppose $p$ is elliptic on supp$(a)$ in the following sense:

$$|p(x,\xi)| \geq cm(x,\xi), \qquad \forall (x,\xi) \in \operatorname{supp} a(\cdot,\hbar), \forall \hbar.$$

  Prove:
    (i) There exits $q_1, q_2 \in S(1/m)$ with $\operatorname{supp}(q_1), \operatorname{supp}(q_2) \Subset \operatorname{supp}(a)$ such that

$$a = q_1 \star p + \hbar^\infty S(1), \quad a = p \star q_2 + \hbar^\infty S(1).$$

    (ii) There exists $C > 0$ such that for all $u \in L^2(\mathbb{R}^n)$,

$$\|\widehat{a}^W u\|_{L^2} \leq C \|\widehat{p}^W u\|_{H_\hbar(1/m)} + O(\hbar^\infty)\|u\|_{L^2}.$$

  (b) Suppose $M$ is compact, $a \in S^0(T^*M)$, $P \in \Psi^k(M)$, and $P$ is elliptic on supp$(a)$ in the following sense:

$$|\sigma(P)| \geq c\langle \xi \rangle^k, \qquad \forall (x,\xi) \in \operatorname{supp} a(\cdot,\hbar), \forall \hbar.$$

    (i) There exits $Q_1, Q_2 \in \Psi^{-k}(M)$ such that

$$\operatorname{Op}(a) = Q_1 \star P + \hbar^\infty \Psi^{-\infty}, \quad \operatorname{Op}(a) = P \star Q_2 + \hbar^\infty \Psi^{-\infty}.$$

    (ii) There exists $C > 0$ such that for all $u \in L^2(M)$,

$$\|\operatorname{Op}(a)u\|_{L^2} \leq C \|\widehat{P}u\|_{H_\hbar^{-k}(M)} + O(\hbar^\infty)\|u\|_{L^2}.$$

(5) [A weak Egorov theorem on manifolds]
Let $(M, g)$ be a compact Riemannian manifold. Suppose $a \in S^{-\infty}(T^*M)$. Let $p(x, \xi) = \|\xi\|_g^2 + V(x)$ and $P = -\hbar^2\Delta + V(x)$, where $V \in C^\infty(M)$. Let $\varphi_t : T^*M \to T^*M$ be the Hamilton flow generated by $p$, and let $a_t = \varphi_t^* a$. Prove: $a_t \in S^{-\infty}(T^*M)$ and

$$\|e^{itP/\hbar} Op(a) e^{-itP/\hbar} - Op(a_t)\|_{\mathcal{L}(L^2(M))} = O(\hbar),$$

where the estimate is uniform for $t \in [0, T]$.
[ Ref: Zworski, *Semiclassical Analysis*. §15.2]

(6) [Sub-principal symbol]
At the end of Lecture 19, we wrote a definition of *sub-principal symbol* for differential operators. Let $P$ be a differential operator of order $k$. Prove $\sigma_{sub}(P)$ is well-defined by checking
(a) $Q = (P - (-1)^k P^*)/2$ is a differential operator of order $k - 1$.
(b) $\sigma_{sub}(P) = \sigma_{k-1}(Q)$.
[Ref: Guillemin-Sternberg, *Semi-classical Analysis*, §1.3.4]