Course Information

♣ About the course:

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  - Yiyu Wang, wangyiyu@mail.ustc.edu.cn
- **Lecture time:** Monday 14:00-15:35
  Tuesday 14:00-16:25
  Thursday 15:55-18:20
- **Webpage:**
  http://staff.ustc.edu.cn/∼wangzuoq/Courses/20S-Topology/index.html
  You may go to the following website to see some old notes:
  http://staff.ustc.edu.cn/∼wangzuoq/Courses/19S-Topology/index.html
- **Homework:**
  Assigned after class, on the bb system.
  Due on Mondays at 11:59 am.
- **Exams:** There will be only one final exam.
- **Your grades:** HWs (35%) + Final (65%)

♢ Notes and Reference books:

Course Notes will be uploaded to the bb system after each lecture.

We will not follow any single book. The following are some nice reference books:

- *Topology, 2nd ed*, by James Munkres
- *Basic Topology*, by M. Armstrong
- *Topology*, by K.Janich

LECTURE 1 — 04/06/2020
INTRODUCTION

1. What is topology

What is topology: some abstract words

According to Bourbaki[1], in mathematics we study structures defined on sets, and maps that preserve structures. Roughly speaking, a structure is a set endowed with some additional feathers on the set, usually prescribed via subsets (of subsets of ...).

There are many mathematical structures, among which the following three structures are most elementary (called *mother structures*):

$$\boxed{\text{algebraic structure}}, \quad \boxed{\text{topological structure}}, \quad \boxed{\text{order structure}}$$

(Other structures: metric structure, measure structure, smooth structure, ...)

Roughly speaking, a topological structure, or a topology for short, defined on a set, is the structure using which one can talk about the conception of *neighborhoods* of en element. As s result, with topology at hand one can talk about the conception of *continuity* for maps defined on such sets.

So topological structures lie in the center of analysis: it is the topology of $\mathbb{R}$ (or $\mathbb{R}^n$) that allows us to talk about the continuity of (multi-variable) functions. Many conceptions and theorems we learned in mathematical analysis are topological, and the main goal of the first half of this course is trying to extend these conceptions and theorems to more general spaces. [2] This part is usually called general topology, or point-set topology.

Topological structures are also most important in geometry: they tell us how points cluster together in a space, without introducing further structures like distance. In particular, you can change the shape by stretching or twisting or bending. Sometimes topology is called "rubber-sheet geometry". In the second half of this course we will focus on geometric properties that are determined by topology.

[1] Bourbaki is the pseudonym of a group of famous mathematicians, including Henri Cartan, Claude Chevalley, Jean Dieudonne, Jean-Pierre Serre, Alexandre Grothendieck, Andre Weil and many others. Bourbaki is most famous for its rigorous presentation of the series of books *Éléments de mathématique* and for introducing the notion of *structures* as the root of mathematics.

[2] Warning: To generalize a theorem $\neq$ to write down the theorem in an abstract setting. Usually you will gain in two aspects: first the new theorem should have more applications, and second the abstract form should be able to help you to understand the nature of the theorem itself.

What is topology: some pictures

As a branch of mathematics, topology is the study of qualitative properties of certain objects (called topological spaces) that are invariant under a certain kind of transformation (called a continuous map).

Let's explain this by some pictures.

- We start with an old joke:

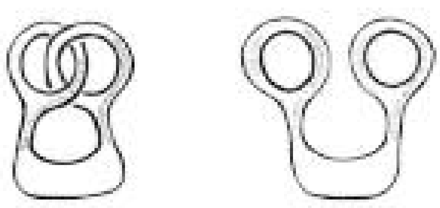
"A topologist is someone who can't tell the different between a coffee cup and a doughnut."

- Topologist always make mistakes when doing computations because...

1=2 ONE=TWO

3≠4 THREE≠FOUR

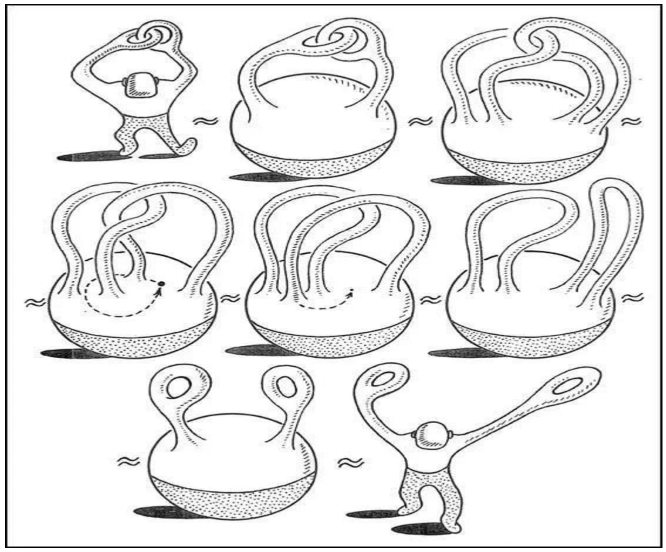
- Here is another interesting problem:



(a) (b)

Two rings are locked together. Can you unlock the two rings without breaking any of them?

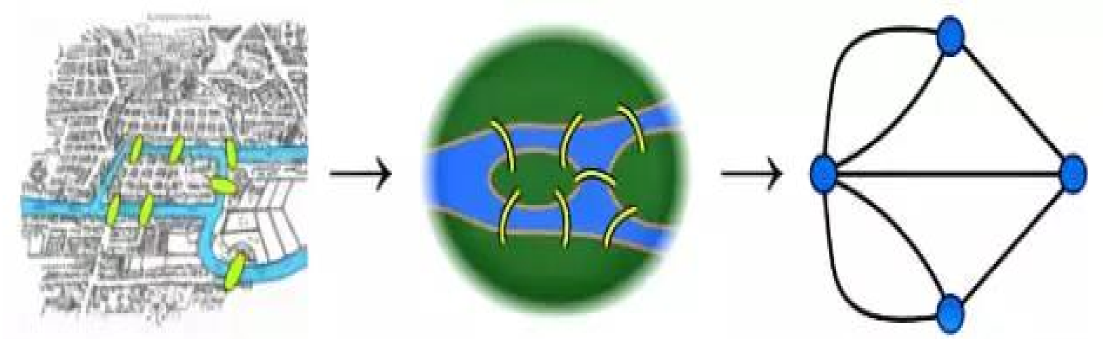
Here is the sol:



• The birth of topology (and graph theory): The Königsberg seven bridge problem.

The city of Königsberg consists of four lands. They are connected by seven bridges. The problem is:

> *Is it possible for a pedestrian to walk across all seven bridges without crossing any bridge twice?*

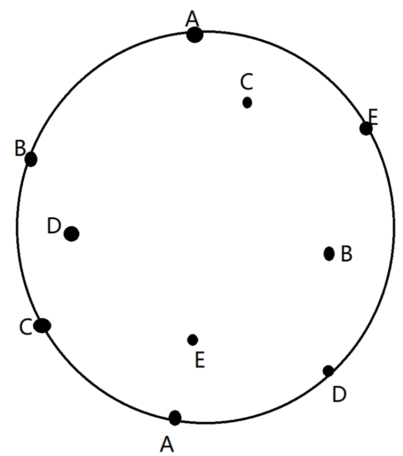


Euler solved this problem negatively in 1736 [3]. The problem did not fit in any existing mathematical framework at that time. It looked like a problem in geometry, but there was no need for distances. Only the relative positions are needed. So we

[3]The true history is interesting: In an earlier letter that Euler wrote to his friend Ehler in 1736, he said "*this type of solution bears little relationship to mathematics....*" Later in another letter Euler wrote in 1736 to Marinoni, he said "*it occurred to me to wonder whether it belonged to the geometry of position, which Leibniz had once so much longed for*". Finally, still in 1736, he solved this problem and generalize his solution to general case.

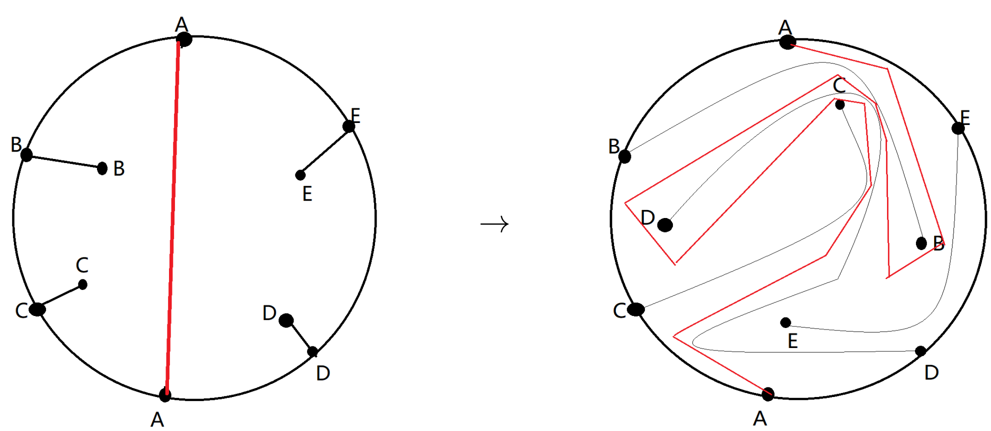
can abstract the situation to get a point-edge graph: four points represent the four lands, while seven edges represent the seven bridges. Can you see why the answer to the Königsberg sever bridge problem is no[4]?

• Here is another example that you can apply the idea of topology to solve problem:



Question: Can you connect points marking the same letter by non-intersecting curves inside the circle?

To solve this problem "topologically", you first continuously move the points inside the circle to positions that you can easily solve the problem. Then continuously move the points back and stretch the curves you have drawn so that they are always non-intersecting:



[4]By the way, in 1875 the city built a new bridge and then the answer is yes.....

2. Some topological theorems that we have learned/will learn

As we mentioned above, topology structure is one of the mother structures in mathematics. In fact, we have used topology in many places in earlier courses. Here we list some of them, together with some related theorems that we will learn later in this course.

(1) We start with a baby theorem:

Baby-Theorem. *Suppose you are given a point A in the upper half plane, and a point B in the lower half plane. If you draw a planar curve connecting A to B, the curve must intersect the x-axis.*

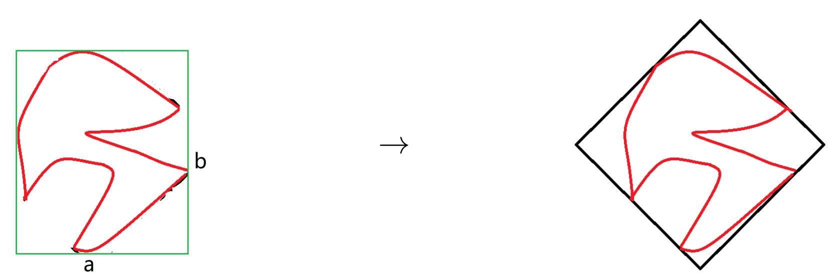
At the very beginning of mathematical analysis, we proved following more general theorem, which is one of the most important theorems in math:

Theorem 2.1 (The Intermediate Value Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then for any value y between $f(a)$ and $f(b)$, there exists $c \in [a,b]$ s.t. $f(c) = y$.*

Topological property behind the IVT: $\boxed{\text{the } \textit{connectedness} \text{ of } [a,b].}$
In this course we will prove:

Theorem 2.2 (The Generalized Intermediate Value Theorem)**.** *Let X, Y be topological spaces, and let $f : X \to Y$ be a continuous map. Then for any connected subset $A \subset X$, the image $f(A)$ is a connected subset in Y.*

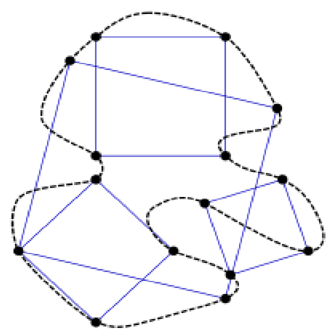
Here is a simple but beautiful application of IVT: Let $\mathcal{C}$ be a closed planar curve. Then there exists a planar square that contains the curve $\mathcal{C}$, so that the curve touches all four sides (vertices count as points in both sides) of the square. To see this you first draw a rectangle satisfying the given property. Then you rotate the rectangle continuously, to get new rectangles (with different side lengths a and b satisfying the same requirement). As you rotate, the value of $a - b$ changes continuously. Moreover, it will change sign after rotating 90 degrees. So it must attain the value 0 during the rotation.



Before we move to next topic, we mention a similar problem in planar geometry. It is proposed by O. Toeplitz in 1911, and is known as the *inscribed square problem*, or the *square peg problem*. It is still unsolved today!

Conjecture (Toeplitz). *On every continuous simple closed planar curve, one can find four points that are the vertices of a square.*

Here is an example on which you can find four such squares:



Remark. (1) For the case of piecewise analytic curves, Emch (1916) proved the conjecture by using some kind of IVT argument.

(2) By using the topology of the Möbius strip, Vaughan gave a very beautiful proof of the following weaker result:

Theorem 2.3. *(Vaughan, 1977) On every continuous simple closed planar curve, one can find four points that are the vertices of a rectangle.*

(2) For the second topic, again we start with a baby theorem:

Baby-Theorem. *Every finite subset of* $\mathbb{R}$ *contains its supremum/infimum.*

In mathematical analysis, we learned

Theorem 2.4 (The Extremal Value Theorem). *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function. Then* f *attains both a maximal value and a minimal value.*

Topological property behind the EVT: $\boxed{\textit{compactness} \text{ of } [a, b].}$

Remarks.
- The theorem fails if you replace $[a, b]$ by (a, b) or $[a, +\infty)$.
- The theorem holds if you replace $[a, b]$ by $[a, b] \cup [c, d]$.
- "compactness is a generalization of finiteness."

We can easily extend the EVT to

Theorem 2.5 (The Generalized Extremal Value Theorem)**.** *Let X, Y be topological spaces, and let $f : X \to Y$ be a continuous map. Then for any compact subset $A \subset X$, the image $f(A)$ is a compact subset in Y.*

Compactness is one of the most important conception in topology. It is widely used in many other branches of mathematics, especially in analysis. Here is some history: In 1856-1857, Dirichlet proposed a method, known as the *Dirichlet't principle*, to solve the equation $\Delta u(x, y, z) = 0$ on a region $\Omega \subset \mathbb{R}^3$ with prescribed boundary value f on the boundary surface $S = \partial\Omega$. To solve this he considered the integral

$$U = \int_\Omega \left[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2 \right] dV.$$

which is always non-negative. Dirichlet concluded that there must be at least one function u on Ω for which the integral reaches a minimum. Then one can show that the minimizer satisfies $\Delta u = 0$. Of course there is a problem: the existence of a greater lower bound for the values of the integral does not necessarily imply the existence of a minimizer. To guarantee this, one need compactness (where we view functions as points). The classical Arzela-Ascoli theorem is one of the first attempts trying to solve this problem. They proved justified Dirichlet's principle under extra assumptions:

Theorem 2.6 (Arzela-Ascoli)**.** *Let $\{f_n\}$ be a uniformly bounded and equicontinuous sequence of real-valued continuous functions defined on $[a, b]$. Then there exists a subsequence that converges uniformly.*

In this course we will prove a much general version:

Theorem 2.7 (Generalized Arzela-Ascoli Theorem)**.** *Let $\mathcal{F}$ be a point-wise bounded and equicontinous family of functions defined on a compact space X. Then for any sequence in $\mathcal{F}$, there exists a subsequence that converges uniformly.*

Arzela-Ascoli theorem is widely used in functional analysis and partial differential equation to prove the existence of a limit. Here we mention another consequence of the Arzela-Ascoli theorem: the *Blaschke selection theorem* in convex geometry, which tells us that any sequence of bounded closed convex sets in any given ball $B(0, R)$ has a subsequence which converges (the meaning of "convergence" will be clear later) to a convex closed set.

(3) Again we start with a baby theorem:

Baby-Theorem. *Suppose $a < b < c < d$. Then one can find a continuous function $f : \mathbb{R} \to \mathbb{R}$ s.t. $f([a, b]) = 0, f([c, d]) = 0$.*

It is not hard to extend this baby theorem to

Theorem 2.8 (Uryson Lemma)**.** *Suppose $K, L \subset \mathbb{R}^n$ are closed sets, and $K \cap L = \emptyset$. Then $\exists$ continuous function $f : \mathbb{R}^n \to \mathbb{R}$ s.t. $f(K) = 0, f(L) = 1$.*

Uryson's lemma is useful because it is equivalent to the following extension theorem:

Theorem 2.9 (Tietze Extension Theorem)**.** *Suppose $K \subset \mathbb{R}^n$ is closed, f is a continuous function defined on K. Then f can be extended to a continuous function on $\mathbb{R}^n$.*

Topological property behind these two Theorems: $\boxed{\textit{separation properties} \text{ of } \mathbb{R}^n}$.

Again we can extend these two theorems to more general topological spaces. For example, we will prove

Theorem 2.10 (Tietze extension theorem)**.** *Suppose X is a normal vector space, and $A \subset X$ is closed. Then any continuous function $f : A \to \mathbb{R}$ can be extended to a continuous function $F : X \to \mathbb{R}$.*

Both compactness and the separation property will also be used to prove the famous Stone-Weierstrass theorem:

Theorem 2.11 (Stone-Weierstrass)**.** *Let X be any compact Hausdorff space, and $L \subset C(X, \mathbb{R})$ be a sub-algebra which vanishes at no point and separate points. Then L is dense.*

(4) Now let's turn to topics in the second half of this course. Again we start with a baby theorem that we learned in mathematical analysis:

Baby-Theorem (Baby fixed point theorem). *Let $f : [a, b] \to [a, b]$ be any continuous function. Then $\exists c \in [a, b]$ s.t. $f(c) = c$.*

In this course we will prove

Theorem 2.12 (Brouwer Fixed Point Theorem for $n = 2$). *Let $\mathbb{D}$ be a planar disc and $f : \mathbb{D} \to \mathbb{D}$ be any continuous map. Then $\exists p \in \mathbb{D}$ s.t. $f(p) = p$.*

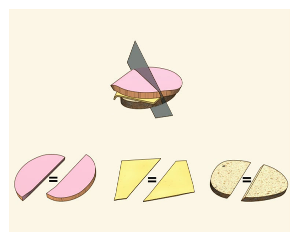
Obviously the fixed point theorem fails if you replace $[a, b]$ by $(a, b]$ or $[a, b] \cup [c, d]$, which indicates the effect of compactness and connectedness. It also fails if you replace $\mathbb{D}$ by a closed planar annulus. The reason is that you can find a circle in the annulus which cannot be deformed to a point. This is a *higher level connectedness*, which is described by the theory of *fundamental groups* (and more generally, *homotopy groups*).

After developing the theory of fundamental groups, we will prove the 2-dimensional Brouwer fixed point theorem as well as the 2-dimensional Borsuk-Ulam theorem:

Theorem 2.13 (Borsuk-Ulam Theorem for $n = 2$). *There does not exists any antipodal continuous map $f : S^2 \to S^1$.*

It has many interesting applications, for example

Theorem 2.14 (The Ham-Sandwich Theorem). *Given any sandwich consisting of bread, cheese and ham, one can cut the sandwich into two pieces by a single knife slice so that each piece has exactly half bread, half cheese and half ham.*



Again these theorems generalize to higher dimension. However, the proofs are more involved: the fundamental groups are not enough. We will not try to develop the general theory of homotopy groups or homology groups to prove these theorems. However, we will prove the general Brouwer fixed point theorem via ideas from *differential topology*:

Theorem 2.15 (Brouwer Fixed Point Theorem for arbitrary n). *Let $\mathbb{B} \subset \mathbb{R}^n$ be a ball and $f : \mathbb{B} \to \mathbb{B}$ be any continuous map. Then $\exists p \in \mathbb{B}$ s.t. $f(p) = p$.*

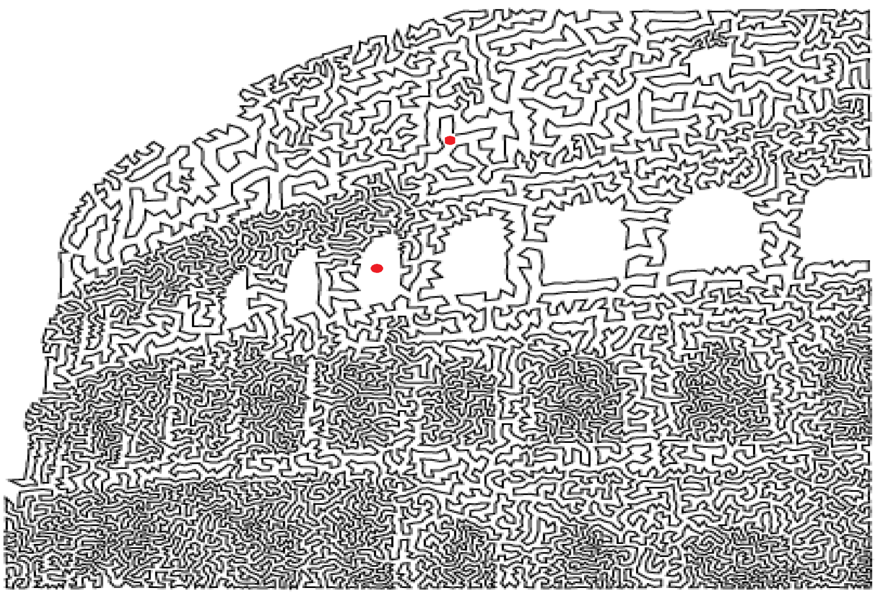
As one application, we will prove the following theorem which "is obviously true":

Theorem 2.16 (Invariance of domain). *If $n \neq m$, then (in topology) $\mathbb{R}^n \neq \mathbb{R}^m$.*

(5) Here is another theorem which "is obviously true" but the proof is very complicated:

Theorem 2.17 (Jordan Curve Theorem)**.** *Any simple closed curve in the plane separate the plane into two disjoint regions.*

To see why the Jordan curve theorem is not that obvious, you can stare at the following picture. It is not quite easy to tell whether the two red dots lie inside or outside the region bounded by the curve!



Remark. Similar theorem holds in higher dimension, which is known as the Jordan-Brouwer separation theorem: *if X is a subset in $\mathbb{R}^n$ which is homeomorphic to S^{n-1}, then $\mathbb{R}^n \setminus X$ has exactly two components, one is bounded and the other is unbounded, so that X is their common boundary.*

(6) Finally let's return to Euler's work. In 1750 Euler proved one of his most famous formula[5]:

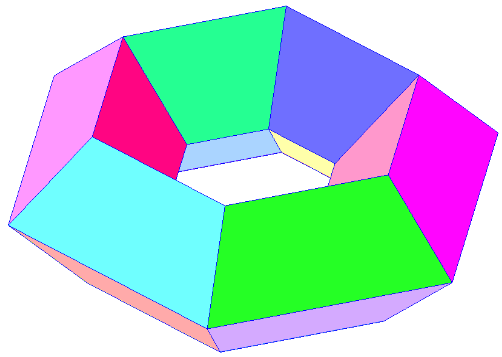
Theorem 2.18 (Euler Polyhedron Formula, 1750)**.** *For any convex polyhedron,*

$$V - E + F = 2,$$

where V, E, F denotes the number of vertices, edges and faces respectively.

You may ask: what about other polyhedrons? For example,

[5] Some people regard this theorem as the birth of topology.



If you count the number of vertices, edges and faces of this torus-like polyhedron, you will find

$$V - E + F = 24 - 48 + 24 = 0.$$

It turns out that the numbers 2 and 0 are topological:

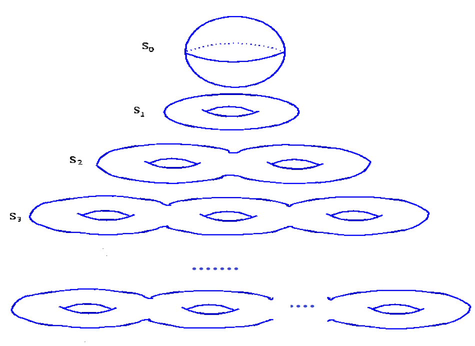
- 2 is the Euler characteristic of S^2 (which is homeomorphic to convex polyhedrons in $\mathbb{R}^3$)
- 0 is the Euler characteristic of the torus T^2.

In general, one has

Theorem 2.19 (Generalized Euler Polyhedron Formula, L'Huilier 1812)**.** *If polyhedron $P \simeq$ surface S, then $V - E + F = \chi(S)$.*

Here, the Euler characteristic $\chi(S)$ of an (oriented) closed surface S is defined to be $2 - 2k$, where k equals "the number of holes" of S. We will prove, at the end of this course, the following classification theorem of surfaces:

Theorem 2.20 (Classification theorem of oriented compact surfaces)**.** *What follows is the list of all oriented compact surfaces without boundary:*



In fact we will prove a more general version: the classification of compact surfaces (including non-oriented surfaces and even compact surfaces with boundary).

Let's end this lengthy introduction by two sayings:

Mathematicians do not study objects, but relations between objects.
– Henri Poincare

"Obvious" is the most dangerous word in mathematics.
– E. Bell