LECTURE 2 — 04/09/2020
METRIC SPACES

Last time we have seen

- Topology is the structure using which one can talk about neighborhood of points and continuity of maps.
- Topology is the underline structure for both analysis and geometry: In analysis we study properties of continuous maps, while in geometry we study properties that are invariant under continuous deformation.

General topology is a very young subject born in the 20th century (while algebraic topology is much older). Historically, Frechet tried to introduce a reasonable conception of an abstract space, and he succeeded by giving the definition of an abstract metric space in his thesis in 1906. After carefully distinguishing the role of neighborhood/limit/distance in metric spaces, Hausdorff was able to give the first definition of topological space (which is not the same as current definitions) in 1912.

1. Metric Spaces

First we recall the definition:

Definition 1.1. A *metric* on a set X is a map

$$d : X \times X \to \mathbb{R}$$

satisfying

(a) (positive definiteness) $d(x, y) \geq 0$, and $d(x, y) = 0 \Leftrightarrow x = y$,
(b) (symmetry) $d(x, y) = d(y, x)$,
(c) (triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$.

The pair (X, d) is called a *metric space*.

Remark. Of course one can remove the assumption $d(x, y) \geq 0$, since it is the consequence of the fact $d(x, x) = 0$, the symmetry and the triangle inequality.

Of course the conception of metric space comes from the metric structure on the Euclidean spaces. However, it turns out that there exists many many metric spaces, in different branches of mathematics. We list some of them:

Example.

(1) On $X = \mathbb{R}$, we have the simplest metric $d(x, y) = |x - y|$.
(2) On $X = \mathbb{R}^n$, we have
   - (the usual Euclidean metric) $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.
   - (the l^1-metric) $d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|$.
   - (the l^∞-metric) $d_\infty(x, y) = \sup\{|x_1 - y_1|, \cdots, |x_n - y_n|\}$.

   They are all special cases of the l^p-metric ($1 \le p \le \infty$):
$$d_p(x, y) = \left(|x_1 - y_1|^p + \cdots + |x_n - y_n|^p\right)^{1/p}.$$
(3) On the infinite Cartesian product
$$X = \mathbb{R}^{\mathbb{N}} = \{(x_1, x_2, \cdots, x_n, \cdots) \mid x_n \in \mathbb{R}\}$$
   we can not define the "l^p metrics" as above because they may diverge. However, one can easily overcome the convergence problem via
   - (the uniform metric)
$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} \min(|x_n - y_n|, 1).$$
   - Here is another very useful metric $\mathbb{R}^{\mathbb{N}}$:
$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$
(4) Another way to solve the convergence problem above is: only consider special subsets. For example,
   - (the l^p space, $1 \le p \le \infty$) Consider the subspace
$$X = l^p(\mathbb{R}) = \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; \|x\|_p := \left(\sum_n |x_n|^p\right)^{1/p} < +\infty \right\} \subset \mathbb{R}^{\mathbb{N}},$$
     then we can define l^p-metric as above:
$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \left(\sum_n |x_n - y_n|^p\right)^{1/p}.$$
   - (the Hilbert cube) Take $X = [0, 1]^{\mathbb{N}} \subset \mathbb{R}^{\mathbb{N}}$, with metric
$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|.$$
(5) (the discrete metric) For any set X, there is a simple metric
$$d(x, y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y. \end{cases}$$

(6) For $X = C([a,b])$ = the set of all continuous functions on $[a,b]$, we have
- (the L^1-metric)
$$d(f,g) = \int_a^b |f(x) - g(x)| dx.$$
- (the L^∞-metric)
$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|.$$
- (the L^2-metric)
$$d(f,g) = \left(\int_a^b |f(x) - g(x)|^2 \mathrm{d}x\right)^{1/2}.$$

They are all special cases of the L^p-metric ($1 \le p \le \infty$):
$$d(f,g) = \left(\int_a^b |f(x) - g(x)|^p \mathrm{d}x\right)^{1/p}.$$

(7) (the $W^{1,2}$-metric) For $X = C^1([a,b])$ = the set of all functions on $[a,b]$ with continuous derivatives, we have the $W^{1,2}$-metric
$$d(f,g) = \left(\int_a^b |f(x) - g(x)|^2 \mathrm{d}x\right)^{1/2} + \left(\int_a^b |f'(x) - g'(x)|^2 \mathrm{d}x\right)^{1/2}.$$

(8) (the word metric) For $X = G$ be a group, we can choose a symmetric generating subset $S \subset G$ [1] and define the word metric induced by S via
$$d(g_1, g_2) = \min\{n : \exists s_1, \cdots, s_n \in S \text{ s.t. } g_1 \cdot s_1 \cdots s_n = g_2\}.$$

(9) (the p-adic metric) Let p be a prime number, and let $X = \mathbb{Q}$ = the set of all rational numbers. Then for any $0 \neq x \in \mathbb{Q}$, we can write
$$x = p^n \frac{r}{s}$$
for some $n, r, s \in \mathbb{Z}$ satisfying $(p,r) = (p,s) = 1$ (uniquely determine n). Define the p-adic norm on $\mathbb{Q}$ by
$$|x|_p = p^{-n} \quad \text{(and set } |0|_p = 0).$$
Then define the p-adic metric on $\mathbb{Q}$ by $d(x_1, x_2) := |x_1 - x_2|_p$.

(10) (The Hausdorff metric) Let X = the set of all bounded closed subsets in $\mathbb{R}$. The Hausdorff metric on X is defined by
$$d(A,B) = \inf\{\varepsilon \ge 0 : A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\},$$
where $A_\varepsilon = \{y \in \mathbb{R} : \exists x \in A \text{ s.t. } |x - y| \le \varepsilon\}$ is the ε-neighborhood of A.

[1] A subset $S \subset G$ is called a generating subset if any element in G can be written as the product of finitely many elements in S. It is called symmetric if $S = S^{-1}$.

On any given metric space, one can easily produce new metrics from old ones:

Proposition 1.2. *Let (X, d) be a metric space. Let $f : [0, +\infty) \to [0, +\infty)$ be a strictly increasing function s.t. $f(0) = 0$ and*

$$f(\alpha + \beta) \leq f(\alpha) + f(\beta), \quad \forall \alpha, \beta \in [0, +\infty).$$

Then $\tilde{d}(x, y) := f(d(x, y))$ is a metric on X.

Proof. It is straightforward to check

- $\tilde{d}(x, y) = 0 \iff d(x, y) = 0 \iff x = y$,
- $\tilde{d}(x, y) = f(d(x, y)) = f(d(y, x)) = \tilde{d}(y, x)$,
- $\tilde{d}(x, z) = f(d(x, z)) \leq f(d(x, y) + d(y, z)) \leq f(d(x, y)) + f(d(y, z)) \leq \tilde{d}(x, y) + \tilde{d}(y, z)$.

□

As a consequence, for any metric space (X, d), the formula

$$\bar{d}(x, y) := \frac{d(x, y)}{1 + d(x, y)}$$

defines a new metric on X. Note: this new metric has the advantage that

$$\bar{d}(x, y) < 1$$

holds for all x, y. In other words, any metric space can be made into a bounded metric space. (We have used this in example (3) above.)

One can also construct new metric spaces from old ones.

Proposition 1.3 ("the subspace metric")**.** *If (X, d) is a metric space and $Y \subset X$ is a subset, then*

$$d_Y := d|_{Y \times Y}$$

is a metric on Y.

Proof. This is quite obvious:

- $d_Y(y_1, y_2) = 0, y_1, y_2 \in Y \subset X \iff y_1 = y_2$.
- $d_Y(y_1, y_2) = d(y_1, y_2) = d(y_2, y_1) = d_Y(y_2, y_1)$.
- $d_Y(y_1, y_3) = d(y_1, y_3) \leq d(y_1, y_2) + d(y_2, y_3) = d_Y(y_1, y_2) + d_Y(y_2, y_3)$.

□

Proposition 1.4 ("the induced metric"). *Let (X, d_1) be a metric space, Y is a set, and $f : Y \to X$ an injective map. Then*

$$d(y_1, y_2) := d_1(f(y_1), f(y_2))$$

is a metric on Y.

Proof. Again

- $d(y_1, y_2) = 0 \iff f(y_1) = f(y_2) \iff y_1 = y_2$.
- $d(y_1, y_2) = d_1(f(y_1), f(y_2)) = d_1(f(y_2), f(y_1)) = d(y_2, y_1)$.
- $d(y_1, y_3) = d_1(f(y_1), f(y_3)) \le d_1(f(y_1), f(y_2)) + d_1(f(y_2), f(y_3)) = d(y_1, y_2) + d(y_2, y_3)$.

□

Proposition 1.5 ("the product metric"). *If $(X, d_1), (Y, d_2)$ are metric spaces, then*

$$d((x_1, y_1), (x_2, y_2)) := d_1(x_1, x_2) + d_2(y_1, y_2)$$

makes $X \times Y$ a metric space.

Proof. We check

- $d((x_1, y_1), (x_2, y_2)) = 0 \iff d_1(x_1, x_2) = 0$ and $d_2(y_1, y_2) = 0 \iff x_1 = x_2, y_1 = y_2$.
- $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2) = d_1(x_2, x_1) + d_2(y_2, y_1) = d((x_2, y_2), (x_1, y_1))$.
- $$\begin{aligned} d((x_1, y_1), (x_3, y_3)) &= d_1(x_1, x_3) + d_2(y_1, y_3) \\ &\le d_1(x_1, x_2) + d_1(x_2, x_3) + d_2(y_1, y_2) + d_2(y_2, y_3) \\ &= d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)). \end{aligned}$$

□

Remark. There are many different ways to put a "product metric" on the Cartesian product of metric spaces. For example, for each $1 \le p \le \infty$, one can define an l^p-type product metric on $X_1 \times \cdots \times X_n$ via the formula

$$d_p((x_1, \cdots, x_n), (y_1, \cdots, y_n)) := \left(|x_1 - y_1|^p + \cdots + |x_n - y_n|^p\right)^{1/p}.$$

By this way, we can regard $(\mathbb{R}^n, d_p)$ as the product of n copies of $(\mathbb{R}, |\cdot|)$.

2. Geometry of metric spaces

As in the Euclidean case, in a metric space (X, d_X) we denote:

- The open ball of radius r centered at x_0:
$$B(x_0, r) = \{x \in X \mid d_X(x, x_0) < r\}.$$
- The closed ball of radius r centered at x_0:
$$\overline{B(x_0, r)} = \{x \in X \mid d_X(x, x_0) \leq r\}.$$
- The sphere of radius r centered at x_0:
$$S(x_0, r) = \{x \in X \mid d_X(x, x_0) = r\}.$$

Given the definitions of balls, we can easily extend the conceptions of open sets and closed sets in the Euclidean case to any metric space:

Definition 2.1 (Open sets and closed sets)**.** Let (X, d) be a metric space.

(1) A subset $U \subset X$ is said to be *open* , if
$$\forall x \in U, \exists \varepsilon = \varepsilon(x) > 0 \text{ s.t. } B(x, \varepsilon) \subset U.$$
(2) A subset $F \subset X$ is said to be *closed* if its complement $F^c = X \setminus F$ is open.

Similarly one can define the conceptions like *the interior points*, *the boundary points* or the *closure* of a set in any metric space.

Example.

(1) Open balls are open, and closed balls are closed.
Note: This is NOT as obvious as it looks like.
— To prove $B(x_0, r)$ is open, by definition, for any $x \in B(x_0, r)$ one need to pick ε carefully, say $\varepsilon = r - d(x, x_0)$, so that $B(x, \varepsilon) \subset B(x_0, r)$.
— To prove $\overline{B(x_0, r)}$ is closed, by definition, for any $x \notin \overline{B(x_0, r)}$ one need to pick ε carefully, say $\varepsilon = d(x, x_0) - r$, so that $B(x, \varepsilon) \subset \overline{B(x_0, r)}^c$.

(2) Two special open sets in any metric space:
   - The total space X itself is always both open and closed.
   - And the empty set $\emptyset$ is always both open and closed.

(3) Let $X = (1, 4) \cup [5, 6] \subset \mathbb{R}$, endowed with the subspace metric d. Then
   - $(1, 4)$ is both open and closed in (X, d). So is $[5, 6]$.
   - $(1, 2)$ is open but not closed in (X, d).
   - $(1, 2]$ is closed but not open in (X, d).
   - $(2, 3]$ is neither open nor closed in (X, d).

(4) Let (X, d) be any set endowed with the discrete metric. Then
   - any subset $A \subset X$ is open, (since $\forall x \in A$, the open ball $B(x, 1) = \{x\} \subset A$.)
   - thus any subset $A \subset X$ is also closed.

We are interested in comparing different metrics.

Question: Can different metrics on the same set generate the same set of open sets?

Answer: Yes.

Example. (1) Open balls in (X, kd) (where $k > 0$) coincide with open balls in (X, d) (with different radii). So kd and d define the same set of open sets.

(2) On $\mathbb{R}^n$, consider the l^1 and l^2 metrics:

$$d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|,$$
$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Then d_1 open balls are not d_2 open balls. However,

$$d_1(x, y) \leq n \cdot \max_i |x_i - y_i| \leq n \cdot d_2(x, y),$$
$$d_2(x, y) \leq \sqrt{n} \cdot \max_i |x_i - y_i| \leq \sqrt{n} \cdot d_1(x, y).$$

It follows

any d_1-open ball contains a smaller d_2-open ball;

any d_2-open ball contains a smaller d_1-open ball.

As a consequence, d_1 open sets are exactly the same as d_2 open sets.

(3) Let (X, d) be any metric space. We have seen that

$$d_1(x, y) := \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X.

Fact: A set $U \subset X$ is d-open if and only if it is d_1-open.

Proof. Since $d_1(x, y) \leq d(x, y)$, we see any d_1-open set is also d-open. (Reason: If U is d_1-open, then $\forall x \in U, \exists \varepsilon > 0$ s.t. $B^{d_1}(x, \varepsilon) \subset U$. But $d_1(x, y) \leq d(x, y) \Rightarrow B^d(x, \varepsilon) \subset B^{d_1}(x, \varepsilon)$. So $B^d(x, \varepsilon) \subset U$.)

Conversely suppose $U \subset X$ is a d-open set. Fix any $x \in U$. Pick $\varepsilon > 0$ such that $B^d(x, \varepsilon) \subset U$. Then for any $y \in B^{d_1}(x, \frac{\varepsilon}{1+\varepsilon})$, we have

$$d_1(x, y) < \frac{\varepsilon}{1 + \varepsilon} \Longrightarrow \frac{d(x, y)}{1 + d(x, y)} < \frac{\varepsilon}{1 + \varepsilon} \Longrightarrow d(x, y) < \varepsilon,$$

where we used the monotonicity of the function $x \mapsto \frac{x}{1+x} = 1 - \frac{1}{1+x}$. As a consequence,

$$B^{d_1}(x, \frac{\varepsilon}{1 + \varepsilon}) \subset B^d(x, \varepsilon) \subset U.$$

It follows by definition that U is also d_1-open. □

Definition 2.2 (equivalent metrics)**.** Let d_1 and d_2 be two metrics on a set X.

(1) We say d_1 and d_2 are *topologically equivalent* if they produce the same set of open sets.
(2) We say d_1 and d_2 are *strongly equivalent* if there exist constants $C_1, C_2 > 0$ s.t.
$$C_1 d_1(x,y) \le d_2(x,y) \le C_2 d_1(x,y), \forall x, y \in X.$$

Remarks.

(1) We don't require d_1-open-balls to be d_2-open-balls!
(2) From the examples we can easily see that strongly equivalent metrics are always topologically equivalent, but the converse is not true: the two metrics in example (3) above are NOT strongly equivalent in general. To see this, one can choose an *unbounded* metric space (X, d)[2]. Since the new metric space (X, d_1) is always bounded, d and d_1 can't be strongly equivalent.

Another useful conception for Euclidean spaces which can be easily extended to abstract metric spaces is the conception of convergence:

Definition 2.3. Let (X, d) be a metric space. We say a sequence of points x_i *converges* to a point x_0 in X (with respect to the metric d) if
$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall i > N, d(x_i, x_0) < \varepsilon.$$

Using the conception of convergence, one can characterize closed sets as in the Euclidean case:

Proposition 2.4. *A subset F in a metric space (X, d) is closed if and only if for any sequence $\{x_n\} \subset F$ with $x_n \to x_0 \in X$, one has $x_0 \in F$.*

Proof. Suppose F is a closed subset in (X, d). Take any sequence $\{x_n\} \subset F$ with $x_n \to x_0 \in X$. To prove $x_0 \in F$, we proceed by contradiction. Suppose $x_0 \notin F$, i.e. $x_0 \in F^c$. Since F^c is open, one can find ε_0 such that $B(x_0, \varepsilon_0) \subset F^c$. By definition of convergence, there exists N such that $d(x_i, x_0) < \varepsilon_0$ for $i > N$. This implies $x_i \in F^c$ for $i > N$, which is a contradiction since we have chosen $x_n \in F$.

Conversely, suppose for any sequence $\{x_n\} \subset F$ with $x_n \to x_0 \in X$, the limit $x_0 \in F$. To prove that F is closed, again we proceed by contradiction. Suppose F is not closed, i.e. F^c is not open. Then there exists $x_0 \in F^c$ so that none of the balls $B(x_0, 1/n)$ is contained in F^c, that is, there exists $x_n \in B(x_0, 1/n)$ with $x_n \notin F^c$, i.e. $x_n \in F$. By the choice of x_n, we have $x_n \to x_0$. So $x_0 \in F$, a contradiction. □

[2]As usual, for a subset A of a metric space (X, d), one can define the diameter of A to be $\text{diam}(A) := \sup\{d(x,y) \mid x, y \in A\}$, which could be $+\infty$. We say A is a bounded subset if $\text{diam}(A) < +\infty$. We say (X, d) is a bounded metric space if $\text{diam}(X) < +\infty$. It is quit obvious that if two metrics on a set are strongly equivalent, then it is bounded with respect to one metric if and only if it is bounded with respect to the other.

3. Continuous maps between metric spaces

We want to define continuity for maps between abstract spaces. For metric spaces, it is easy to do so since we have defined the conception of convergence:

Definition 3.1 (continuous map)**.** Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is *continuous* at $x_0 \in X$ if for any sequence x_i which converges to x_0 in X, the image $f(x_i)$ converges to $f(x_0)$ in Y. We say the map f is a *continuous map* if it is continuous at every $x_0 \in X$.

It is not hard to prove

Lemma 3.2. *A map $f : (X, d_X) \to (Y, d_Y)$ is* continuous *at $x_0 \in X$ if and only if*

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, d_X(x, x_0) < \delta \Longrightarrow d_Y(f(x), f(x_0)) < \varepsilon.$$

To get a better understanding the meaning of the continuity in metric spaces, let's gover some simple examples.

Example.

(1) For Euclidean spaces, continuity is the same as what we have learned in calculus.
(2) Let (X, d) be any metric space.
- for any fixed $\bar{x} \in X$, the function

  $$d_{\bar{x}} : X \to \mathbb{R}, x \mapsto d_{\bar{x}}(y) := d(x, \bar{x})$$

  is continuous (where we always endow $\mathbb{R}$ with the usual metric).

  Proof. For $\forall \varepsilon > 0, \forall x_0 \in X$, and $\forall x \in X$ with $d(x, x_0) < \varepsilon$, we have

  $$|d_{\bar{x}}(x) - d_{\bar{x}}(x_0)| = |d(x, \bar{x}) - d(x_0, \bar{x})| \le d(x, x_0) < \varepsilon.$$

  [So d_X is in fact *Lipschitz continuous* with Lipschitz constant 1.] □

- More generally, for any $A \subset X$, we can define

  $$d_A : X \to \mathbb{R},\ x \mapsto d_A(x) := \inf\{d(x, y) : y \in A\}.$$

  Fact: d_A is continuous.

  Sketch of proof. First apply the triangle inequality to prove

  $$|d_A(x) - d_A(y)| \le d(x, y).$$

  Then the conclusion follows. □

- If we endow $X \times X$ with the "product metric" $d_{X \times X}$ as in Proposition 1.5, then the function $d : X \times X \to \mathbb{R}$ is continuous. [Try to prove this.]

(3) Endow the space $X = C([a,b])$ with the metric
$$d_X(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|.$$
Then the "integration map"
$$\int : X \to \mathbb{R},\ f \mapsto \int_a^b f(x)\mathrm{d}x$$
is continuous, since
$$\left|\int_a^b f(x)\mathrm{d}x - \int_a^b g(x)\mathrm{d}x\right| \le \int_a^b |f(x) - g(x)|\mathrm{d}x \le (b-a)\cdot d_X(f,g).$$

(4) Let X be any set, and d_X be the discrete metric on X. Let (Y, d_Y) be any metric space.
- Any map $f : X \to Y$ is continuous.

  Proof. For any $\varepsilon > 0$, we simply take $\delta = 1$. Then for any $x, x_0 \in X$ with $d_X(x, x_0) < 1$, we must have $x = x_0$ since d_X is the discrete metric. It follows $d_Y(f(x), f(x_0)) = 0 < \varepsilon$. □

- Among all maps $f : Y \to X$, only "locally constant maps" are continuous.

  Proof. [A map $f : Y \to X$ is locally constant means: for any $y_0 \in Y$, there is an $\delta > 0$ so that $f(y) = f(y_0)$ for all y satisfying $d(y_0, y) < \delta$.]
  Obviously if f is locally constant, then it is continuous.
  Conversely, suppose $f : Y \to X$ is continuous at y_0. Then there exists $\delta > 0$ such that for any $y \in Y$ with $d_Y(y, y_0) < \delta$, we have $d_X(f(y), f(y_0)) < 1$, which implies $f(y) = f(y_0)$ since d_X is the discrete metric. □

We can rewrite Lemma 3.2 as:

Lemma 3.3 (the geometric interpretation of continuity at a point)**.**
A map $f : (X, d_X) \to (Y, d_Y)$ is continuous at a point x_0 if and only if it maps the "δ-neighborhood of x_0" into an "ε-neighborhood of $f(x_0)$", i.e.
$$\forall \varepsilon > 0, \exists \delta > 0 \ s.t.\ f(B_X(x_0, \delta)) \subset B_Y(f(x_0), \varepsilon).$$

This can be further rewritten via open sets:

Proposition 3.4. *Let $f : (X, d_X) \to (Y, d_Y)$ be a map between metric spaces. Then f is continuous at $x \in X$ if and only if for any open set $V \subset Y$ with $f(x) \in V$, the preimage $f^{-1}(V)$* contains *an open set $U \subset X$ with $x \in U$.*

Remark. Note: We don't claim here that $f^{-1}(V)$ is open in X!

Proof. Suppose f is continuous at $x \in X$, and $V \subset Y$ is open such that $f(x) \in V$. By definition, $\exists \varepsilon > 0$ s.t. $B(f(x), \varepsilon) \subset V$. By continuity of f at x, $\exists \delta > 0$ such that $f(B(x,\delta)) \subset B(f(x),\varepsilon)$. So $B(x,\delta) \subset f^{-1}(B(f(x),\varepsilon)) \subset f^{-1}(V)$.

Conversely suppose for any open set $V \subset Y$ with $f(x) \in V$, $f^{-1}(V)$ contains an open set $U \ni x$. Then in particular for $\forall \varepsilon > 0, f^{-1}(B(f(x),\varepsilon))$ contains an open set U with $x \in U$. By the definition of open set, $\exists \delta > 0$ s.t. $B(x,\delta) \subset U$, which implies $B(x,\delta) \subset f^{-1}(B(f(x),\varepsilon))$. So $f(B(f(x),\delta)) \subset B(f(x),\varepsilon)$, i.e. f is continuous at x. □

As a consequence, we get the following characterization of continuous maps between abstract metric spaces:

Theorem 3.5. *A map $f : (X, d_X) \to (Y, d_Y)$ is continuous if and only if for any open set V in Y, the preimage $f^{-1}(V)$ is open in X.*

Proof. Suppose f is continuous, and $V \subset Y$ is open. Then $\forall x \in f^{-1}(V)$, by Proposition 3.4, $f^{-1}(V)$ contains an open set U with $x \in U$. So $f^{-1}(V)$ is open in X.

Conversely suppose for any open set $V \subset Y$, the preimage $f^{-1}(V)$ is open in X. For any $x \in X$, take any open set V in Y with $f(x) \in V$. Then $f^{-1}(V)$ itself is a open set in X which contains the point x. So by the Proposition above, f is continuous. □

As a consequence, we have

Corollary 3.6. *If $f : (X, d_X) \to (Y, d_Y)$ is continuous, $\tilde{d_X}$ and $\tilde{d_Y}$ are metrics topologically equivalent to d_X and d_Y respectively, then $f : (X, \tilde{d_X}) \to (Y, \tilde{d_Y})$ is continuous.*

In conclusion:

> Although we defined continuity via the metric structure, continuity is really independent of the metric: it depends only on the collection of open sets produced by the metric!

So continuity is a topological property, not a metric property.

To compare, let's end today's lecture with a similar conception: the "uniform continuity" of a map. The definition is straightforward:

Definition 3.7. A map $f : (X, d_X) \to (Y, d_Y)$ is *uniformly continuous* if

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Of course uniformly continuous functions are continuous, but the converse is not true. It turns out that "uniform continuity" is NOT a topological property: it does depend on the metric.

Example. Let d be the standard metric on $\mathbb{R}$, and let d_1 be the metric on $\mathbb{R}$ induced by the map $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$, i.e.

$$d_1(x, y) := |\arctan(x) - \arctan(y)|.$$

Then open balls of d_1 are exactly open intervals in $\mathbb{R}$. So d and d_1 induces the same set of open sets, i.e. they are topologically equivalent.

Consider the identity map

$$f : \mathbb{R} \to \mathbb{R}, f(x) = x.$$

Then $f : (\mathbb{R}, d) \to (\mathbb{R}, d)$ is uniformly continuous, but $f : (\mathbb{R}, d_1) \to (\mathbb{R}, d)$ is NOT uniformly continuous since

$$d_1(n, n+1) = |\arctan(n) - \arctan(n+1)| \to 0 \text{ as } n \to \infty$$

but $d(n, n+1) = 1$.