Topology (H) Lecture 7
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COMPACTNESS: TYCHONOFF THEOREM AND ITS APPLICATIONS

Last time we learned:

- Compact, sequentially compact and limit point compact
- The image of compact/sequentially compact sets under continuous maps are still compact/sequentially compact.
- Closed subsets of compact space are still compact. Finite product of compact spaces is still compact.

Today we will prove one of the most surprising, important and useful theorems in general topology, which was first proved in a special case in 1930 by A.N. Tychnoff in 1930, who stated the general version in 1935[1]:

Theorem 0.1 (Tychonoff Theorem (1935)).
If X_α is compact for each α, then the product $(\prod_\alpha X_\alpha, \mathscr{T}_{product})$ is compact.

At the first glance, it seems that the theorem is unlikely to be true. Compactness should behave like "finite". How could an infinite or even an uncountable product be compact?!

However: let's look at a couple examples from which we can see compactness. Recall that

$$X^{\mathbb{N}} = \prod_{n\in\mathbb{N}} X = \{(a_1, a_2, \cdots) \mid a_i \in \mathbb{N}\}.$$

As we have seen, the product topology on $X^{\mathbb{N}}$ can be identified with the pointwise convergence topology on $\mathcal{M}(\mathbb{N}, X)$.

(a) $X = \{0, 2\}$ the "two point set" $\rightsquigarrow$ $X^{\mathbb{N}}$: sequences of 0's and 2's.
Observation: Each such sequence $\rightsquigarrow$ a nested sequence of closed sets in the construction of Cantor set. (Or alternatively, each sequence, when viewed as the digits of the ternary representation of a real number in $[0, 1]$.)
Fact: $X^{\mathbb{N}}$ is homeomorphic to the Cantor set C, which is compact! (exercise)
(b) $X = [0, 1]$ Element in $X^{\mathbb{N}}$ are sequences $a = (a_1, a_2, \cdots)$, $a_i \in [0, 1]$ By using diagonalization trick one can prove $X^{\mathbb{N}}$ is sequentially compact! (PSet 3-1-2(c))

[1] In fact, Tychonoff defined the product topology in his 1935 paper for the first time.

1. Different ways to characterize compactness

Before we prove Tychonoff theorem, we need some preparation.

First it is not surprise that by applying the "open-closed duality", we can convert "the definition of compact sets via open sets" to an equivalent definition via closed sets:

$X = \bigcup_\alpha U_\alpha$, U_α open $\Rightarrow \exists U_{\alpha_i}, X=\bigcup_{i=1}^k U_{\alpha_i}$.	$\Leftrightarrow$	$\emptyset = \bigcap_\alpha F_\alpha$, F_α closed $\Rightarrow \exists F_{\alpha_i}$, $\emptyset=\bigcap_{i=1}^k F_{\alpha_i}$.	$\Leftrightarrow$	$\bigcap_{i=1}^k F_{\alpha_i} \neq \emptyset$ for any finite collection $\{F_{\alpha_1}, \cdots, F_{\alpha_k}\}$ $\Rightarrow \bigcap_\alpha F_\alpha \neq \emptyset$.

So we arrive at

Proposition 1.1 (Characterize compactness via closed sets). *A topological space X is compact if and only if it satisfies the following property:*

> *[Finite Intersection Property] If $\mathscr{F} = \{F_\alpha\}$ is any collection of closed sets s.t. any finite intersection*
> $$F_{\alpha_1} \cap \cdots \cap F_{\alpha_k} \neq \emptyset,$$
> *then $\cap_\alpha F_\alpha \neq \emptyset$.*

As a consequence, we get

Corollary 1.2 (Nested closed sets). *Let X be compact, and*
$$X \supset F_1 \supset F_2 \supset \cdots$$
be a nested sequence of closed sets. Then $\bigcap_{n=1}^\infty F_n \neq \emptyset$.

It is NOT surprising that we can characterize compactness via "base covering":

Proposition 1.3. *Let $\mathcal{B}$ be a base of $(X, \mathscr{T})$. Then X is compact if and only if any base covering $\mathscr{U} \subset \mathcal{B}$ of X, one can find a finite sub-covering.*

Proof. Suppose X is compact, and let $\mathscr{U} \subset \mathcal{B}$ be any base covering. Since $\mathcal{B} \subset \mathscr{T}$, $\mathscr{U}$ is automatically an open covering. So it admits a finite sub-covering.

Conversely suppose any base covering of X admits a finite sub-covering, and let $\mathscr{U}$ be any open covering of X. For any $x \in X$, there exists $U^x \in \mathscr{U}$ and $U_x \in \mathcal{B}$ s.t.
$$x \in U_x \subset U^x.$$
Since $\{U_x\}$ is a base covering of X, there exist $U_{x_1}, \cdots, U_{x_m}$ s.t. $X = \bigcup_{i=1}^n U_{x_i}$. It follows that for $U^{x_1}, \cdots, U^{x_n} \in \mathscr{U}$, $X = \bigcup_{i=1}^n U^{x_i}$, so X is compact. □

It is natural to extend this proposition to sub-base.

Theorem 1.4 (Alexander sub-base theorem). *Let $\mathcal{S}$ be a sub-base of $(X, \mathscr{T})$ Then X is compact if and only if any "sub-base covering" of X has a finite sub-covering.*

Surprisingly, the proof is much more harder: we need the axiom of choice!

Axiom of Choice. *Let X be a set, and let $\mathscr{A}$ be a collection of non-empty subsets of X. Then there exists a choice function $f : \mathscr{A} \to X$, i.e. a function such that $f(A) \in A$ for all $A \in \mathscr{A}$.*

In mathematics, "axiom of choice" looks like a monster. On one hand, using the axiom of choice, one gets many nice results, e.g. one can

- construct a Hamel basis for any vector space,
- prove the Hahn-Banach theorem (to construct special linear functionals).

On the other hand, one also gets counterintuitive results: one can

- construct non-measurable sets;
- decompose the 3-dimensional ball $B^3 \subset \mathbb{R}^3$ into finitely many pieces, and after using only rotations and translations, resemble these pieces into two copies of B^3 (Banach-Tarski paradox)

According to Russell,

> "To choose one sock from each of infinitely many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed. "

There are many equivalent ways to state the axiom of choice. Some of them are horrible, and some of them looks "obviously true", for example:

An equivalent statment of A.C.. *If $X_\alpha \neq \emptyset$ for $\forall\, \alpha$, then $\prod_\alpha X_\alpha \neq \emptyset$.*

Two other widely used equivalent formulation of A.C are "well-ordering theorem" and "Zorn's lemma". To state them, we need

Recall. For a non-empty partially order set $(\mathcal{P}, \preceq)$,

- a nonempty subset $\mathcal{Q} \subset \mathcal{P}$ is *linear ordered* if $\forall\, a, b \in \mathcal{Q} \Rightarrow a \preceq b$ or $b \preceq a$;
- an element $c \in \mathcal{P}$ is an *upper bound* for $\mathcal{Q} \subset \mathcal{P}$ if $a \preceq c$ for $\forall\, a \in \mathcal{Q}$;
- an element $c \in \mathcal{P}$ is *maximal* if $\forall\, b \in \mathcal{P}$, if $c \preceq b$, then $b = c$.
- $(P, \preceq)$ is *totally ordered* if for $\forall a, b \in P$, either $a \preceq b$ or $b \preceq a$.
- $(P, \preceq)$ is *well ordered* if it is totally ordered, and any non-empty subset in P admits a minimal element.

Well-ordering theorem. *Any set can be make into a well-ordered set.*

Zorn's lemma. *Let $(\mathcal{P}, \preceq)$ be a non-empty partially order set s.t. every linearly ordered subset of $\mathcal{P}$ has an upper bound in $\mathcal{P}$, then $\mathcal{P}$ contains at least one maximal element.*

According to Jerry Bona,

> "The axiom of choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?"

Proof of Alexander Sub-base Theorem.

Proof. We only need to prove: *If any sub-basic covering of X has a finite sub-covering, then X is compact.* The other half is obvious.

We prove by contradiction. So we suppose X is NOT compact, but any sub-basic covering of X has a finite sub-covering. Let

$\boxed{科}$[2] $= \{\mathscr{A} \subset \mathscr{T} \mid \mathscr{A}$ is an open covering of X, but has no finite sub-covering$\} \in 2^{2^{2^X}}$.

Then

- Since X is non-compact, $\boxed{科} \neq \emptyset$.
- $\boxed{科}$ is a partially ordered set w.r.t. the set inclusion relation.
- If ㊖[3] $\subset \boxed{科}$ is a linearly ordered subset, then
  (1) $\mathscr{E} = \bigcup_{\mathscr{A} \in ㊖} \mathscr{A} \subset \mathscr{T}$,
  (2) $\mathscr{E}$ is an open covering of X,
  (3) $\mathscr{E}$ is an upper bound for ㊖.
  If fact, $\mathscr{E} \in \boxed{科}$. If NOT, then $\mathscr{E}$ has a finite sub-covering $\{A_1, A_2, \cdots, A_n\}$. By construction, $\exists\, \mathscr{A}_1, \cdots, \mathscr{A}_n \in$ ㊖ s.t. $A_i \in \mathscr{A}_i$. Since ㊖ is a linearly ordered set, $\exists\, k \in \{1, 2, \cdots, n\}$ s.t.
  $$\mathscr{A}_i \preceq \mathscr{A}_k,\ \forall\, i \in \{1, 2, \cdots, n\}.$$
  It follows that $A_1, \cdots, A_n \in \mathscr{A}_k$, i.e. $\mathscr{A}_k$ has a finite sub-covering, which is a contradiction.

So by Zorn's lemma, $\boxed{科}$ has a maximal element $\mathscr{A}$.

Now let's use the sub-base $\mathcal{S}$. We claim:

Claim 1. *$\mathcal{S} \cap \mathscr{A}$ is an open covering of X.*

First let's suppose this is true. In other words, we get a covering $\mathcal{S} \cap \mathscr{A}$ of X. This covering can't have finite sub-covering since $\mathscr{A}$ has no finite sub-covering. But on the other hand, it must has a finite sub-covering since it is a sub-basic covering. Contradiction! This completes the proof.

□

[2]This is a new character whose pronunciation is *wǒ kē*. In this course, we

- use lower case letters like a, b, x etc to represent elements in X;
- use capital letters like A, B, U etc to represent subsets in X, i.e. elements in 2^X;
- use script font $\mathscr{A}$, $\mathscr{T}$ etc to represents collection of subsets in X, i.e. elements in 2^{2^X};
- create characters to represent collections of collections of subsets in X, i.e. elements in $2^{2^{2^X}}$.

[3]Pronunciation: *nǐ kē*

Proof of Claim 1. For any $x \in X$, $\exists\, A \in \mathscr{A}$ s.t. $x \in A$. By the definition of sub-base, $\exists\, S_1, \cdots, S_m \in \mathcal{S}$ s.t. $x \in S_1 \cap \cdots \cap S_m \subset A$. We want to show that

$$\exists\, 1 \le k \le m \text{ s.t. } S_k \in \mathscr{A}.$$

(This implies $S_k \in \mathcal{S} \cap \mathscr{A}$ and $x \in S_k$, so we are done.)

Again by contradiction. If NOT, Then for $\forall\, 1 \le k \le m$,

$$\mathscr{A}_k := \mathscr{A} \cup \{S_k\} \succ \mathscr{A}.$$

Since $\mathscr{A}$ is a maximal element of $\boxed{科}$, we must have $\mathscr{A}_k \notin \boxed{科}$, i.e. $\mathscr{A}_k$ has a finite sub-covering $\{S_k, A_{k,1}, \cdots, A_{k,j(k)}\}$, where $A_{k,j} \in \mathscr{A}$. It follows that

$$X = \bigcap_{k=1}^{m} (S_k \cup A_{k,1} \cup \cdots \cup A_{k,j(k)}) = (S_1 \cap \cdots \cap S_m) \cup (\bigcup_{k,j} A_{k,j}),$$

where we used the fact $(A \cup B) \cap (C \cup D) \subset (A \cap C) \cup B \cup D$. As a consequence,

$$\{A, A_{k,j} \mid 1 \le k \le m, 1 \le j \le j(k)\}$$

is a finite sub-covering of $\mathscr{A}$. Contradiction! □

2. Proof of Tychonoff Theorem

By definition, the product topology $\mathscr{T}_{product}$ on $\prod_\alpha X_\alpha$ is the topology generated by the sub-base

$$\mathcal{S} = \cup_\alpha \{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \subset X_\alpha \text{ is open}\},$$

where $\pi_\alpha : \prod_\beta X_\beta \to X_\alpha$ is the standard projection. So it is "natural" way to prove Tychonoff theorem using Alexander sub-base theorem.

Proof of Tychonoff theorem.

Let $\mathscr{A}$ be any sub-basic covering of $X = \prod_\alpha X_\alpha$. In other words, $\mathscr{A}$ has the form

$$\mathscr{A} = \{\pi_\alpha^{-1}(U) \mid U \in \mathscr{A}_\alpha\}$$

where $\mathscr{A}_\alpha \subset \mathscr{T}_\alpha$ is a collection of open sets in X_α. Since $\mathscr{A}$ is a covering of $X = \prod_\alpha X_\alpha$, there exists α_0 s.t. $\mathscr{A}_{\alpha_0}$ is a covering of X_{α_0}, otherwise [4]

$$\begin{aligned} \forall\, \alpha, X_\alpha \setminus \textstyle\bigcup_{U \in \mathscr{A}_\alpha} U \neq \emptyset &\implies \prod_\alpha (X_\alpha \setminus \textstyle\bigcup_{U \in \mathcal{A}_\alpha} U) \neq \emptyset \\ &\implies \mathscr{A} \text{ is NOT a covering of } X! \end{aligned}$$

Now by the compactness of X_{α_0}, $\mathscr{A}_{\alpha_0}$ has a finite sub-covering $\{U_1, \ldots, U_m\}$. It follows $\{\pi_\alpha^{-1}(U_1), \cdots, \pi_\alpha^{-1}(U_m)\}$ is a finite sub-covering of $\mathscr{A}$. So by Alexander's sub-base theorem, X is compact. □

[4]Note: Here we use Axiom of Choice again!

Remark.

(1) In PSet 3-1-2, we have seen that the product of countably many sequentially compact spaces (endowed with the product topology) is still sequentially compact, but the product of uncountably many sequentially compact spaces, like $[0,1]^{[0,1]}$, fails to be sequentially compact w.r.t. the product topology.

(2) It is easy to see that in general, the product of infinitely many spaces fails to be compact when endowed with the box topology. We have seen the failure for sequential compactness.

(3) According to Tychonoff theorem, $([0,1]^{[0,1]}, \mathscr{T}_{product})$ is compact. Thus

Corollary 2.1. *Compact $\not\Rightarrow$ sequentially compact.*

(4) Conversely, let A be the subset of $\mathcal{M}([0,1],\mathbb{R})$ consisting of those functions that are non-zero at only countably many points, (c.f. Lecture 5, page 5)

$$A = \{f : [0,1] \to \mathbb{R} \mid f(x) \neq 0 \text{ for countably many } x \in [0,1]\}.$$

Then A is sequentially compact with respect to the pointwise convergence topology: any sequence $\{f_n\}$ in A must has a convergent subsequence, since the set $S = \{x \mid \exists n \text{ s.t. } f_n(x) \neq 0\}$ is a countable set, and the pointwise convergence of a subsequence of f_n on $[0,1]$ is equivalent to the pointwise convergence of the same sequence on S. Thus we can apply the sequentially compactness for countably product of sequentially compact subspaces. However, A is not compact, since for any $t \in [0,1]$, if we denote

$$A_t := \{f \in A \mid f(t) = 1\},$$

then $\{A_t\}$ is a collection of closed sets in A which violates the finite intersection property: $\cap_{t\in[0,1]} A_t = \emptyset$, while $\cap_{i=1}^k A_{t_i} \neq \emptyset$. Thus

Corollary 2.2. *Sequentially compact $\not\Rightarrow$ compact.*

(5) In fact, Kychonoff theorem is equivalent to the axiom of choice since

Proposition 2.3 (Kelley: Tychonoff Theorem $\Rightarrow$ Axiom of Choice)**.**
Suppose Tychonoff Theorem is true, then (the equivalent statement of A.C.)

$$X_\alpha \neq \emptyset, \forall\, \alpha \implies \prod_\alpha X_\alpha \neq \emptyset.$$

Proof. Each X_α is compact w.r.t. the trivial topology. Define $\widetilde{X_\alpha} = X_\alpha \cup \{\infty\}$, with the topology $\widetilde{\mathscr{T}_\alpha} = \{\emptyset,\, X_\alpha,\, \{\infty_\alpha\},\, \widetilde{X_\alpha}\}$ which is still compact. By Tychonoff theorem, $X = \prod_\alpha \widetilde{X_\alpha}$ is compact w.r.t. the product topology.
Observation: $\{\pi_\alpha^{-1}(X_\alpha)\}$ is a family of closed sets in X with F.I.P. (Why?)
If follows from compactness that

$$\bigcap_\alpha \pi_\alpha^{-1}(X_\alpha) \neq \emptyset.$$

By definition, any element in $\cap_\alpha \pi_\alpha^{-1}(X_\alpha)$ is an element in $\prod_\alpha X_\alpha$. ◻

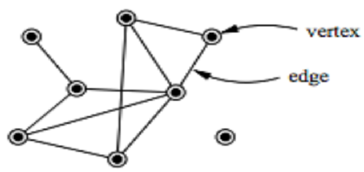
3. Applications of Tychonoff Theorem

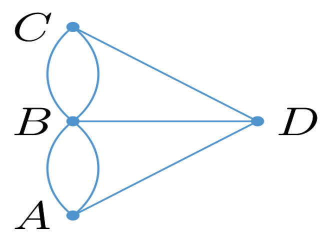
We shall give several applications of Tychonoff theorem.

3.1. Application 1: Graph Coloring.

Definition 3.1. A *graph* G is a pair $G = (V, E)$, where

- V is a set, whose elements are called *vertices*.
- $E \subset V \times V$, whose elements are called *edges*. (could be a "multi-set")





Definition 3.2. Let $G = (V, E)$ be a graph.

- A *subgraph* $\widetilde{G}$ of G is a graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ s.t. $\widetilde{V} = V$, $\widetilde{E} \subset E$. [5]
- A subgraph $\widetilde{G}$ is a *finite subgraph* if $\widetilde{E}$ is a finite set.
- A *coloring* of G by k colors is a map
$$f : V \to [k] := \{1, 2, \cdots, k\}$$
s.t. for any edge $\overline{ab} \in E$, one has $f(a) \neq f(b)$

Theorem 3.3 (de Bruijn-Erdös)**.** *Let G be any graph (where V could be infinite or even uncountable), and $k \in \mathbb{N}$. If any finite subgraph of G is k-colorable, then G is k-colorable.*

Proof. Endow $[k] = \{1, 2, \cdots, k\}$ with the discrete topology and consider the product space
$$X := \prod_V [k] = \{f : V \to \{1, 2, \cdots, k\}\}.$$
Since $[k]$ is compact, Tychonoff Theorem implies X is compact. For any subset $F \subset E$, we define
$$X_F := \{f : V \to [k] \mid f \text{ is a coloring of } (V, F)\}.$$

[5]Usually for a subgraph one only requires $\widetilde{V} \subset V$ instead of $\widetilde{V} = V$. But for our purpose below, it is equivalent and more convenient if we take $\widetilde{V} = V$.

Fact 1 If $F = \{\overline{ab}\}$ is a one-edge set, then X_F is closed.

$$\begin{aligned}\text{Reason: } X_{\{\overline{ab}\}} &= \{f : V \to [k] \mid f(a) \neq f(b)\} \\ &= \bigcup_{i \neq j} \{f : V \to [k] \mid f(a) = i, f(b) = j\} \\ &= \bigcup_{i \neq j} (\pi_a^{-1}(i) \cap \pi_b^{-1}(j))\end{aligned}$$

is a finite union of closed sets.

Fact 2 For any $F \subset E$, X_F is closed.

$$\text{Reason: } X_{F_1} \cap X_{F_2} = X_{F_1 \cup F_2} \implies X_F = \bigcap_{\overline{ab} \in F} X_{\overline{ab}}$$

is an intersection of closed sets.

Now we consider the following collection of closed sets

$$\mathscr{F} = \{X_F \mid F \subset E \text{ is a finite set}\}.$$

We check that $\mathscr{F}$ satisfies F.I.P:

$$X_{F_1} \cap X_{F_2} \cap \cdots \cap X_{F_n} = X_{F_1 \cup \cdots \cup F_m} \neq \emptyset.$$

(Since $F_1 \cup \cdots \cup F_m$ is finite!) Since X is compact, we get

$$\bigcap_{X_F \in \mathscr{F}} X_F \neq \emptyset.$$

But by definition, any element $f \in \cap_{X_F \in \mathscr{F}} X_F$ is a k-coloring of G. □

3.2. Application 2: Arithmetic progressions in subsets in $\mathbb{Z}$.

We can also use topology to study number theory.

Definition 3.4. A *partition* of $\mathbb{Z}$ is a decomposition

$$\mathbb{Z} = S_1 \cup \cdots \cup S_k$$

s.t. $S_i \cap S_j = \emptyset$ for $i \neq j$. [Equivalently, it is an k-coloring of $\mathbb{Z}$ (with no edges).]

Theorem 3.5 (Van der Waerden 1927). *For any partition $\mathbb{Z} = S_1 \cup \cdots \cup S_k$, $\exists j$ s.t. S_j contains arbitrary long (but still finite) arithmetic progressions.*

Van der Waerden's theorem is a precursor of the famous Szemerendi's theorem:

Theorem 3.6 (Szemeredi, 1975). *Any subset $A \subset \mathbb{N}$ with*

$$\limsup_{n \to \infty} \frac{|A \cap \{1, \cdots, n\}|}{n} > 0$$

contains infinitely many arithmetic progressions of length k for all positive integers k.

In 1977, Furstenberg[6] gave an ergodic theory reformulation and obtained a proof using topology, and then proved a multidimensional generalization of Szemeredi's theorem. In what follows we state a topological version of van der Waerden's theorem, using which we prove the number-theoretic version.

Theorem 3.7 (Topological version of van der Waerden)**.** *Let X be compact, $T : X \to X$ be a homeomorphism, and $\{V_\alpha\}$ an open covering of X. Then $\forall\, l \in \mathbb{N}$, $\exists\, n \in \mathbb{N}$ and open set $V \in \{V_\alpha\}$ s.t.*

$$V \cap T^{-n}V \cap \cdots \cap T^{-(l-1)n}V \neq \emptyset.$$

Theorem 3.7 can be proven in the framework of general topology after introducing some conceptions in the subject "topological dynamical system", but we will omit that. In what follows we prove:

Theorem 3.7 ⇒ Theorem 3.5

Proof. Again endow $[k]$ with the discrete topology. By Tychonoff, the space

$$\widetilde{X} = \prod_{\mathbb{Z}}[k] = \{f : \mathbb{Z} \to [k]\}$$

is compact. Note that any partition of $\mathbb{Z}$ corresponds to an element in $\widetilde{X}$.

Let $T : \widetilde{X} \to \widetilde{X}$ be the "right-shift" map: $T(f)(n) = f(n-1)$. Then T is continuous since $T^{-1}(\pi_n^{-1}(i)) = \pi_{n-1}^{-1}(i)$, and $\{\pi_n^{-1}(i)\}$ is a sub-base. Similarly T^{-1}, the "left-shift" map, is continuous. So T is a homeomorphism.

Let $f \in \widetilde{X}$ be the element that corresponds to the partition in Theorem 3.5. Let

$$X = \overline{\{T^n f \mid n \in \mathbb{Z}\}} = \overline{\{\cdots, T^{-2}f, T^{-1}f, f, Tf, T^2 f, \cdots\}}$$

As a closed set in the compact spact $\widetilde{X}$, X itself is compact. Moreover, $T(X) = X$:

Reason: $T(X) \supset \{T^n f \mid n \in \mathbb{Z}\} \Rightarrow T(X) \supset X$.
Replace T with T^{-1}, we get $T^{-1}(X) \supset X$.

So $T : X \to X$ is a homeomorphism. (Think of this: why?)

Now for each $i \in [k]$, denote

$$V_i = \{f \in \widetilde{X} \mid f(0) = i\} = \pi_0^{-1}(i).$$

Then V_i's are open in $\widetilde{X}$, and form an open covering of X. By Theorem 3.7, $\forall\, l \in \mathbb{N}$, $\exists\, n \in \mathbb{N}$ and $V_j \in \{V_i \mid 1 \leq i \leq k\}$ s.t.

$$(V_j \cap T^{-n}V_j \cap \cdots \cap T^{-(l-1)n}V_j) \cap X \neq \emptyset.$$

[6]H. Fursterberg is a famous American-Israeli mathematician who won Wolf prize in 2006/7 and won Abel prize in 2020, for his "pioneering the use of methods from probability and dynamics in group theory, number theory and combinatorics". You have seen his proof of the infinitude of prime numbers in PSet 2-1-2: the proof was published in 1955 when he was still an undergraduate student in Yeshiva University.

Note: $V_j \cap T^{-n}V_j \cap \cdots \cap T^{-(l-1)n}V_j$ is open in $\widetilde{X}$, while $X = \overline{\{T^{-n}f \mid n \in \mathbb{Z}\}}$. So there exists m s.t.

$$T^{-m}f \in V_j \cap T^{-n}V_j \cap \cdots \cap T^{-(l-1)n}V_j,$$

i.e.

$$f \in T^m V_j \cap T^{m-n}V_j \cap \cdots \cap T^{m-(l-1)n}V_j.$$

It follows

$$f(m) = f(m-n) = \cdots = f(m-(l-1)n) = j.$$

In other words, $m, m-n, \cdots, m-(l-1)n \in S_j$. □

3.3. **Application 3: The Banach-Alaoglu theorem.**

The third application is to functional analysis. Recall that a *normed vector space* is a vector space X endowed with a norm structure, i.e. a function $\|\cdot\| : X \to [0, +\infty)$ such that for any $x, y \in X$ and any $\lambda \in \mathbb{C}$,

- $\|x\| \geq 0$, and $\|x\| = 0 \Leftrightarrow x = 0$;
- $\|x+y\| \leq \|x\| + \|y\|$;
- $\|\lambda x\| = |\lambda|\|x\|$.

On any normed vector space $(X, \|\cdot\|)$, one can easily check that $d(x,y) := \|x-y\|$ defines a metric structure. So we can always endow X with the metric topology, and talk about continuity of maps. In particular, we denote

$$X^* := \{l : X \to \mathbb{C} \mid l \text{ is complex linear and continuous}\}.$$

The space X^* is again a linear space, on which one can define a norm

$$\|l\| := \sup_{\|x\|=1} |l(x)|.$$

The new normed vector space $(X^*, \|\cdot\|)$ is called the *dual space* of $(X, \|\cdot\|)$. It is again automatically a metric space and we can talk about conceptions like the closed unit ball,

$$\overline{B^*} := \{l \in X^* \mid \|l\| \leq 1\}.$$

Unfortunately, in most applications, normed vector spaces and their dual spaces are infinitely dimensional, and thus the closed unit ball are not compact with respect to the usual metric topology.

Of course the reason for the non-compactness is that the metric topologies are too strong, i.e. they contain too much open sets. In Lecture 4 (page 8, example 2) we introduced two new topologies: the weak topology on X and the weak-$*$ topology on X^*. The weak topology on X is the weakest topology making all linear functionals $l \in X^*$ continuous, and the weak-$*$ topology on X^* is the weakest topology making all evaluation maps ev_x continuous. From this definition it is quite easy to see that the weak-$*$ topology is the pointwise convergence topology if we regard X^* as a subset of $\mathcal{M}(X, \mathbb{C})$. Since the pointwise convergence topology on $\mathcal{M}(X, \mathbb{C})$ can be identified with the product topology on $\mathbb{C}^X$, it is not too surprising that the closed unit ball $\overline{B^*}$ is closed with respec to the weak-$*$ topology:

Theorem 3.8 (Banach-Alaoglu)**.** *Let X be a normed vector space. Then the closed unit ball $\overline{B^*}$ in the dual space X^* is compact with respect to the weak-$*$ topology.*

Proof. The idea is to identify the closed unit ball $\overline{B^*}$ of X^* with a closed subset of the product space

$$Z = \prod_{x\in X} \{z \in \mathbb{C} \mid |z| \le \|x\|\} \subset \mathbb{C}^X,$$

where Z is compact since we endow Z with the product topology.

As we explained above, we can identify X^* with a subspace of $\mathcal{M}(X, \mathbb{C})$ and thus any element $l \in X^*$ satisfying $\|l\| \le 1$ can be identified with an element in Z. Conversely, an element f in Z belongs to the closed unit ball $\overline{B^*}$ in X^* if and only if

$$f(x + y) = f(x) + f(y) \qquad \text{and} \qquad f(\lambda x) = \lambda f(x)$$

for all $x, y \in X$ and $\lambda \in \mathbb{C}$. In other words, $f \in \overline{B^*}$ if and only if, as an element of Z, it belongs to the set

$$\begin{aligned} D &= \{f \in Z \mid \mathrm{ev}_{x+y}(f) = \mathrm{ev}_x(f) + \mathrm{ev}_y(f), \quad \mathrm{ev}_{\alpha x}(f) = \alpha \mathrm{ev}_x(f), \forall x, y \in X, \forall \alpha \in \mathbb{C}\} \\ &= \bigcap_{x,y,\alpha} (\mathrm{ev}_{x+y} - \mathrm{ev}_x - \mathrm{ev}_y)^{-1}(0) \cap (\mathrm{ev}_{\alpha x} - \alpha \mathrm{ev}_x)^{-1}(0). \end{aligned}$$

By continuity of the evaluation maps, D is a closed subset in Z. Since Z is compact, so is D. One can carefully check that the identification above is a homeomorphism between $(\overline{B^*}, \mathscr{T}_{\text{weak}*})$ and $(D, \mathscr{T}_{producut})$. So $\overline{B^*}$ is compact with respect to the weak-$*$ topology. ☐

Remark. Under suitable extra assumptions one can prove $\overline{B^*}$ is also weak-$*$ sequentially compact. This fact is very useful in analysis, especially in P.D.E.