Topology (H) Lecture 9
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AXIOMS OF COUNTABILITY

Last time we learned

- Topological properties of metric spaces: (T2), (T4), (A1)
- Metric properties of metric spaces: bounded, totally bounded, complete = absolute closed, Lebesgue number
- Equivalent characterizations of compactness in metric spaces: compact = sequentially compact = limit point compact = complete and totally bounded

1. Axioms of countability

Let's recall

Definition 1.1. A topological space $(X, \mathscr{T})$ is called *first countable*, or an *(A1)-space*, if it satisfies

$$\text{(A1)} \qquad \begin{array}{l} \text{For any } x \in X, \text{ there exists a countable family of open} \\ \text{neighborhoods of } x, \{U_1^x, U_2^x, U_3^x, \cdots\}, \text{ such that for any} \\ \text{open neighborhood } U \text{ of } x, \text{ there exists } n \text{ s.t. } U_n^x \subset U. \end{array}$$

Such a family $\{U_n^x \mid n \in \mathbb{N}\}$ is called a *countable neighbourhood base* at x.

Remark. If $(X, \mathscr{T})$ is first countable, then for each point one can choose a countable neighborhood base $\{U_n^x\}$ satisfying

$$U_1^x \supset U_2^x \supset U_3^x \supset \cdots,$$

since if one has a countable neighbourhood base $V_1^x, V_2^x, \cdots$ at x, then one can take

$$U_1^x = V_1^x, \;\; U_2^x = V_1^x \cap V_2^x, \;\; U_3^x = V_1^x \cap V_2^x \cap V_3^x, \; \cdots.$$

It is easy to check that $\{U_n^x | n \in \mathbb{N}\}$ is a decreasing neighbourhood base at x.

Last time we have seen

Proposition 1.2. *Suppose $(X, \mathscr{T})$ is first countable.*

(1) A subset $F \subset X$ is closed if and only if it contains all its sequential limits, i.e. for any sequence $\{x_n\} \subset F$ with $x_n \to x \in X$, we have $x \in F$.
   - *As a consequence: a map $f : X \to Y$ is continuous if and only if it is sequentially continuous, i.e. if $x_n \to x_0$, then $f(x_n) \to f(x_0)$.*

(2) If $(X, \mathscr{T})$ is also Hausdorff, then a subset A in X is limit point compact if and only if it is sequentially compact.

Here are some examples of (A1) spaces and non-(A1) spaces.

Example.

(1) Any metric space is first countable since we can take
$$U_n^x = B(x, \frac{1}{n}).$$
(2) The Sorgenfrey line $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is first countable since we can take
$$U_n^x = [x, x + \frac{1}{n}).$$
However, we will see soon that this space is NOT metrizable.

(3) The space $(\mathbb{R}, \mathscr{T}_{cocountable})$ is NOT first countable: For any sequence of open neighborhoods $\{U_x^n\}$ of x, one can always construct a new open set by dropping of one more point from $\cap_n U_x^n$, which cannot contain any U_x^n.

(4) The space $(\mathcal{M}([0,1], \mathbb{R}) = \mathbb{R}^{[0,1]}, \mathscr{T}_{product})$ is NOT first countable, since we have seen (in Lecture 5 page 5) that there exists a non-closed subset
$$A = \{f : [0,1] \to \mathbb{R} \mid f(x) \neq 0 \text{ for only countably many } x\}$$
which contains all its sequential limit points.

For Euclidean space $\mathbb{R}^n$, we have seen in Lecture 4 that not only it has a countable neighborhood base at each point x, but it has a base which contains only countably many open sets,
$$\mathcal{B} = \{B(x, r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}_{>0}\}.$$
This is a stronger countability property, which deserve a name:

Definition 1.3. A topological space $(X, \mathscr{T})$ is called *second countable*, or an *(A2)-space*, if it satisfies

(A2) there exists a countable family of open sets $\{U_1, U_2, U_3, \cdots\}$ which form a base of the topology $\mathscr{T}$.

Such a family $\{U_n \mid n \in \mathbb{N}\}$ is called a *countable base* of $\mathscr{T}$.

Obviously any second countable space is a first countable space. But the converse is not true, for example, $(\mathbb{R}, \mathscr{T}_{discrete})$ is first countable as it is a metric space, but it is not second countable.

There is a big class of metric spaces which is second countable:

Proposition 1.4. *Any totally bounded metric space is second countable.*

Proof. Suppose (X, d) is totally bounded. By definition, for any n, one has a finite $\frac{1}{n}$-net, i.e. there exists finitely many points $x_{n,1}, x_{n,2}, \cdots, x_{n,k(n)} \in X$ such that
$$X = \bigcup_{i=1}^{k(n)} B(x_i, \frac{1}{n}).$$

We claim that the countable collection

$$\mathcal{B} := \{B(x_{n,i}, 1/n) : n \in \mathbb{N}, 1 \leqslant i \leqslant k(n)\}$$

is a base of the metric topology $\mathscr{T}$. To see this, we take any open set U and any point $x \in U$. Then there exists $\varepsilon > 0$ s.t. $B(x, \varepsilon) \subset U$. Now we choose $n \in \mathbb{N}$ and $1 \leq i \leq k(n)$ s.t.

$$\frac{1}{n} < \frac{\varepsilon}{2} \quad \text{and} \quad d(x, x_{n,i}) < \frac{1}{n}.$$

It follows

$$B(x_{n,i}, \frac{1}{n}) \subset B(x, \frac{2}{n}) \subset B(x, \varepsilon) \subset U,$$

i.e. the countable family $\mathcal{B}$ is a base. □

Since any compact metric space is totally bounded, we get as a consequence,

Any compact metric space is second countable.

So for example, $[0, 1]$ is second countable.

Example. Consider the Hilbert cube

$$X = [0, 1]^{\mathbb{N}} = \{(x_1, x_2, \cdots) \mid x_i \in [0, 1]\}.$$

It is a metric space with the metric (Lecture 2, page 2)

$$d((x_n), (y_n)) := \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|.$$

So on X we have a metric topology. By construction it also admits a product topology.
Claim: The metric topology $\mathscr{T}_{metric}$ and the product topology $\mathscr{T}_{product}$ on X coincide.

Proof. Take any metric ball $B((x_n), r)$. Take N_0 large and ε_0 small s.t.

$$2^{-N_0} < \frac{r}{2} \quad \text{and} \quad \varepsilon_0 < \frac{r}{2N_0}.$$

Then for any point $(y_n) \in \prod_{i=1}^{N_0} (x_i - \varepsilon_0, x_i + \varepsilon_0) \times \prod_{i>N_0} [0, 1]$, we have

$$d((x_n), (y_n)) < \sum_{i=1}^{N_0} 2^{-i} \varepsilon_0 + \sum_{i>N_0} 2^{-i} < N_0 \varepsilon_0 + 2^{-N_0} < r.$$

It follows that any metric ball contains a smaller open set in the product topology. Since metric balls generate $\mathscr{T}_{metric}$, we conclude that $\mathscr{T}_{product}$ is stronger than $\mathscr{T}_{metric}$.

Conversely, for any $\mathscr{T}_{product}$-open set of the form

$$U = \prod_{i=1}^{N_0} (x_i - \varepsilon_0, x_i + \varepsilon_0) \times \prod_{i>N_0} [0, 1]$$

which generates $\mathscr{T}_{product}$, we can take $r = 2^{-N_0} \varepsilon_0$ and it is easy to check $B((x_n), r) \subset U$. So $\mathscr{T}_{metric}$ is also stronger than $\mathscr{T}_{product}$. □

As a consequence, we see that the Hilbert cube $(X, \mathscr{T}_{product})$ is a compact metric space, and thus is second countable.

Definition 1.5. We say a topological space $(X, \mathscr{T})$ is *metrizable* if there exists a metric structure on X so that the metric topology coincides with $\mathscr{T}$.

Remark. According to what we proved last time, any metrizable topological space must be first countable, Hausdorff and normal.

On the other hand, if we endow X with the box topology, then $(X, \mathscr{T}_{box})$ is not second countable. In fact, we have
Fact: $(X, \mathscr{T}_{box})$ is not even first countable.

Proof. Fix any point $x = (x_i)$ in X. Suppose on the contrary that

$$\{U_n(x) = \prod_i U_i^{(n)}(x_i) \mid n \in \mathbb{N}\}$$

is a countable neighborhood base of $(X, \mathscr{T}_{box})$ at (x_i). Note that each $U_i^{(n)}(x_i)$ is an open neighborhood of x_i in $[0,1]$. Let $\widetilde{U}_i^{(n)}(x_i) \subsetneqq U_i^{(n)}(x_i)$ be a strictly smaller open neighborhood of x_i in $[0,1]$. Then the set

$$U := \prod_i \widetilde{U}_i^{(i)}(x_i)$$

is an open neighborhood of the point (x_n) in the box topology, but none of the $U_n(x)$'s is contained in U, a contradiction. □

It follows that $([0,1]^{\mathbb{N}}, \mathscr{T}_{box})$ is not metrizable.

If you think very hard about the countable base for the Euclidean space $\mathbb{R}^n$ that we constructed, namely

$$\mathcal{B} = \{B(x,r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}_{>0}\},$$

you will find that a crucial reason is that $\mathbb{R}^n$ admits a countable dense subset $\mathbb{Q}^n$. It turns out that this is the common feather for any second countable space,

Proposition 1.6. *Any second countable topological space has a countable dense subset.*

Proof. Let $\{U_n | n \in \mathbb{N}\}$ be a countable base of $(X, \mathscr{T})$. For each n, we choose a point $x_n \in U_n$. Let

$$A = \{x_n | n \in \mathbb{N}\}.$$

Then A is a countable subset in X. We claim that $\overline{A} = X$. In fact, for any $x \in X$ and any open neighborhood U of x, there exists n s.t. $x \in U_n \subset U$. In particular, $U \cap A \neq \emptyset$. So we get $\overline{A} = X$. □

Remark. What we really proved is: In any topological space, there exists a dense subset whose cardinality is no more than the cardinality of a base.

Definition 1.7. A topological space $(X, \mathscr{T})$ is *separable* if it contains a countable dense subset.

So we can rewrite the proposition we just proved as

Any second countable topological space is separable.

Remarks.

(1) The converse is NOT true. For example,
Fact: $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is separable, but not second countable.
Proof. To show $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is separable, it is enough to show $\overline{\mathbb{Q}} = \mathbb{R}$ with respect to the Sorgenfrey topology, which follows from the fact that for any $x \in \mathbb{R}$ and any interval $[x, x+\varepsilon)$, one can find a rational number $r \in [x, x+\varepsilon)$.
To see $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is not second countable, we let $\mathcal{B}$ be any base of $\mathscr{T}_{sorgenfrey}$. Then for any $x \in \mathbb{R}$, there exists an open set $B_x \in \mathcal{B}$ s.t.

$$x \in B_x \subset [x, x+1),$$

which implies $x = \inf B_x$. As a consequence, for any $x \neq y$, we have $B_x \neq B_y$. So $\mathcal{B}$ is an uncountable family. □
However, we have

Proposition 1.8. *A metric space (or a metrizable topological space) is second countable if and only if it is separable.*

Proof. Let (X, d) be a separable metric space, and $A = \{x_n | n \in \mathbb{N}\}$ be a countable dense subset. Then

$$\mathcal{B} = \{B(x_n, \frac{1}{m}) | n, m \in \mathbb{N}\}$$

is a countable base for the metric topology. □

Corollary 1.9. $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ *is NOT metrizable.*

(2) Separability is a very useful conception in functional analysis. It is used to prove certain compactness results. Another well-known result is
A Hilbert space $\mathcal{H}$ is separable if and only if it has a countable orthonormal basis.
From this fact it is easy to construct non-separable Hilbert spaces. For example, consider

$$\widetilde{l^2(\mathbb{R})} = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) \neq 0 \text{ for countably many } x, \text{ and } \sum_x |f(x)|^2 < \infty\}.$$

It is an inner product space with inner product

$$\langle f, g \rangle := \sum_{x \in \mathbb{R}} f(x)g(x),$$

which induces a metric structure and admits a completion. The resulting Hilbert space can't admit any countable orthogonal basis.

Roughly speaking, separability means you can use countably many data to "recover" the whole space. Here is an example:

Proposition 1.10. *Any compact metric space* (X, d) *is (topologically) homeomorphic to a closed subset of the Hilbert cube* $[0,1]^{\mathbb{N}}$.

Proof. Since X is compact, it is bounded. By scaling the metric d, we may assume $\mathrm{diam}(X) \leqslant 1$. Let $A = \{x_n | n \in \mathbb{N}\}$ be a countable dense subset in X. We define

$$\begin{aligned} F : X &\to [0,1]^{\mathbb{N}} \\ x &\mapsto (d(x, x_1), d(x, x_2), \cdots, d(x, x_n), \cdots). \end{aligned}$$

Then we have:

- F is continuous since each $\pi_n \circ F = d(x, x_n)$ is continuous.
- F is injective: if $F(x) = F(y)$, then

$$d(x, x_n) = d(y, x_n), \quad \forall n.$$

  Since A is dense, there exists $x_{n_k} \to x$. By continuity of d,

$$d(x, y) = \lim_{k\to\infty} d(x_{n_k}, y) = \lim_{k\to\infty} d(x_{n_k}, x) = 0.$$

- $[0,1]^{\mathbb{N}}$ is Hausdorff since it is a metric space.

It follows [see PSet 3-3-1(a)] that the map

$$F : X \to F(X) \subset [0,1]^{\mathbb{N}}$$

is a homeomorphism. Obviously, $F(X)$ is closed, since it is "a compact subset in a Hausdorff space". □

Remark. Countability is always combined with compactness. There are some other conceptions which has a mixed flavor of countability and compactness, e.g .

Definition 1.11. A topological space $(X, \mathscr{T})$ is *Lindelöf* if any open covering $\mathscr{U}$ of X admits a countable sub-covering.

Definition 1.12. A topological space $(X, \mathscr{T})$ is *countably compact* if every countable open covering of X admits a finite sub-covering.

We will not study them here, instead we will leave some properties as exercises.

2. Urysohn's metrization theorem

We have seen some necessary conditions for a topological space to be metrizable: It must be first countable, Hausdorff and normal. On the other hand, these conditions are not sufficient. For example,
Fact: The Sorgenfrey line $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is first countable, Hausdorff, normal but not metrizable.

Proof. We have seen that $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is first countable but not metrizable. Obviously it is Hausdorff, since any $x < y$ can be separated by open sets $[x, (x+y)/2)$ and $[y, y+1)$. It remains to show that $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$ is normal, i.e. disjoint closed sets can be separated by disjoint open sets. So we let A, B be disjoint closed sets. For any $a \in A$, we have $a \in B^c$. Since B^c is open, we can take $\varepsilon_a > 0$ such that $[a, a+\varepsilon_a) \cap B = \emptyset$. Similarly for any $b \in B$ we take $\varepsilon_b > 0$ such that $[b, b+\varepsilon_b) \cap A = \emptyset$. Note that we always has

$$[a, a+\varepsilon_a) \cap [b, b+\varepsilon_b) = \emptyset, \quad \forall a \in A \text{ and } b \in B,$$

otherwise we will have $b \in [a, a+\varepsilon_a)$ or $a \in [b, b+\varepsilon_b)$, which is a contradiction. It follows that

$$U_A := \cup_{a \in A}[a, a+\varepsilon_a) \quad \text{and} \quad U_B := \cup_{b \in B}[b, b+\varepsilon_b)$$

are disjoint open sets separating A and B. □

Although the metrization problem is subtle in general, it has a very simple answer for second countable spaces.

Theorem 2.1 (Urysohn's metrization theorem)**.** *A second countable topological space $(X, \mathscr{T})$ is metrizable if and only if it is Hausdorff and normal.*

Remark.

- The proof is again "from countably many information to recover the whole". However, we can't replace the assumption "second countable" by "separable", as can be seen by the counterexample $(\mathbb{R}, \mathscr{T}_{sorgenfrey})$.
- The proof is similar to that of Proposition 1.10: We will embed $(X, \mathscr{T})$ into the Hilbert cube $[0, 1]^{\mathbb{N}}$, so that $(X, \mathscr{T})$ inherits a subspace metric! The only issue is: we don't use countably many points to separate. Instead, we need to make use of the countable base.

Our major tool in the proof is the following very important result of Urysohn [1] which we will prove next time:

Lemma 2.2 (Urysohn's Lemma)**.** *A topological space* $(X, \mathscr{T})$ *is normal if and only if for any disjoint closed sets* $A, B \subset X$*, there exists a continuous function* $f : X \to [0,1]$ *such that* $f^{-1}(0) \supset A, f^{-1}(1) \supset B$*.*

Proof of Urysohn's Metrization Theorem.

Step 1. As we mentioned above, we want to construct an "embedding"

$$F : X \to [0,1]^{\mathbb{N}}.$$

To do so, we let $\mathcal{B} = \{B_n | n \in \mathbb{N}\}$ be a countable base of $\mathscr{T}$. Consider

$$I = \{(m,n) | \overline{B_n} \subset B_m\} \subset \mathbb{N} \times \mathbb{N}.$$

Claim: $I \neq \emptyset$.

Proof of Claim. For any $x \in X$ and any open neighborhood U of x, since $\{x\}$ and U^c are disjoint closed sets in X, by the definition of normal space, there exists open neighborhood U_1 of x and open neighborhood V_1 of U^c such that

$$U_1 \cap V_1 = \emptyset.$$

Since $\mathcal{B}$ is a base, there exists $B_m \in \mathcal{B}$ s.t.

$$x \in B_m \subset U_1.$$

Similarly since $\{x\}$ and B_m^c are disjoint closed sets, there exists disjoint open sets U_2, V_2 s.t.

$$x \in U_2, \quad \text{and} \quad B_m^c \subset V_2.$$

Now take $B_n \in \mathcal{B}$ s.t.

$$x \in B_n \subset U_2.$$

Then

$$\overline{B_n} \subset \overline{U_2} \subset V_2^c \subset (B_m^c)^c = B_m.$$

□

In fact we proved more:

For any $x \in U$, there exists $(m,n) \in I$ s.t. $x \in B_n$ and $U^c \subset B_m^c$.

[1] P. Urysohn was a famous Soviet mathematician. He was awarded his habilitation with topic "integral equations" at Moscow University in June 1921, and turned to topology after that. In about three years he made a big contribution to dimension theory, and proved many important and fundamental theorems including Urysohn lemma and Urysohn metrization theorem. He died in 1924, at age 26, while swimming off the coast of Brittany, France.

Step 2. Now for any $(m,n) \in I$, by applying Urysohn's lemma to the pair of disjoint closed sets $\overline{B_n}$ and B_m^c, we can find a continuous function $g_{n,m} : X \to [0,1]$ s.t.

$$g_{n,m}(\overline{B_n}) = 1 \quad \text{and} \quad g_{n,m}(B_m^c) = 0.$$

Since I is a countable set, we will renumber $g_{n,m}$'s to $f_1, f_2, f_3, \cdots$. Note that the sequence of functions $f_1, f_2, \cdots$ satisfy the property that

$$\boxed{\forall x \in X \text{ and any open } U \ni x, \text{ there exists } n \text{ s. t. } f_n(x) = 1, f_n(U^c) = 0.}$$

Step 3. Finally we define

$$F : X \to [0,1]^{\mathbb{N}}, \quad x \mapsto (f_1(x), f_2(x), \cdots).$$

Claim: F is a homeomorphism from X to $F(X)$ in $[0,1]^N$.

Proof of Claim. Since each f_i is continuous, F is continuous. Moreover, F is injective since for any $x \neq y$, there exists n s.t.

$$f_n(x) = 1 \quad \text{and} \quad f_n(y) = 0.$$

So F is a continuous and bijective map from X to $F(X)$. To prove F is a homeomorphism onto its image, we only need to prove F is an open map onto its image $F(X)$.

We let $U \subset X$ be open, and $z_0 \in F(U)$. Take $x_0 \in U$ such that $z_0 = F(x_0)$. Take n s.t.

$$f_n(x_0) > 0 \quad \text{and} \quad f_n(U^c) = 0.$$

Let $V = \pi_n^{-1}((0,+\infty))$, where π_n is the projection map from $[0,1]^{\mathbb{N}}$ to its n^{th} component. Then V is open in $[0,1]^{\mathbb{N}}$. So

$$W := V \cap F(X)$$

is open in $F(X)$.

Sub-Claim: $z_0 \in W \subset F(U)$.

Proof of Sub-Claim. We have $z_0 \in W$ since

$$\pi_n(z_0) = \pi_n(F(x_0)) = f_n(x_0) > 0.$$

We have $W \subset F(U)$ since for any $z \in W$, there exists x s.t.

$$F(x) = z \quad \text{and} \quad f_n(x) > 0,$$

which implies that $x \in U$ and thus $z \in F(U)$. □

As a consequence, $F(U)$ is open in $F(X)$. So the Claim is proved. □

Since $F(X)$ is a subset in metric space $[0,1]^{\mathbb{N}}$, it admits a subspace metric whose topology is the same as the subspace topology. Now pull-back the metric to X. It is obvious that the resulting metric topology on X coincides with the original topology on X. □