SEPARATION AXIOMS

Last time we learned

- Axioms of countability: (A1), (A2), separable, Lindelöf
  - For \((X, \mathcal{T})\), (A2) \(\implies\) (A1), Separable, Lindelöf,
  - For \((X, d)\), (A2) \(\iff\) Separable \(\iff\) Lindelöf.

- A useful (counter)example: \((\mathbb{R}, \mathcal{T}_{Sorgenfrey})\)
  - (A1), (T2), (T4), separable, Lindelöf, NOT (A2), NOT metrizable.

- Metrization of topological spaces:
  - Necessary conditions: (A1), (T2), (T4)
  - Hilbert cube \([0, 1]^\mathbb{N}, \mathcal{T}_{prod}\) is metrizable
  - Urysohn’s metrization theorem: (A2)+(T2)+(T4) \(\implies\) metrizable.

1. Separation Axioms

By “separation axioms” we means properties of topological spaces concerning separating certain disjoint sets via (disjoint) open sets. [Caution: It is very different from the conception separable that we learned last time!] There are many different separations axioms \(^1\), four of them are used more often than the others, and we have seen two of them which are most important:

\[\forall x_1 \neq x_2 \in X, \exists \text{ open sets } U, V \text{ s.t.} \]
\[x_1 \in U \setminus V \text{ and } x_2 \in V \setminus U.\]

\[\forall x_1 \neq x_2 \in X, \exists \text{ open sets } U, V \text{ s.t.} \]
\[x_1 \in U, x_2 \in V \text{ and } U \cap V = \emptyset.\]

\[\forall \text{ closed sets } A \text{ and } x \notin A, \exists \text{ open sets } U, V \text{ s.t.} \]
\[A \subset U, x \in V \text{ and } U \cap V = \emptyset.\]

\[\forall \text{ closed sets } A \text{ and } B \text{ with } A \cap B = \emptyset, \exists \text{ open sets} \]
\[U, V \text{ s.t. } A \subset U, B \subset V \text{ and } U \cap V = \emptyset.\]

\(^1\)In literature there are at least 20 different separation axioms. According to Wikipedia, “the history of the separation axioms in general topology has been convoluted, with many meanings competing for the same terms and many terms competing for the same concept”.

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**Remark.** In different books, “regular”, “(T3)”, “normal”, “(T4)” could have very different meanings. For example, in some books “regular” or “(T3)” means “both (T1) and (T3)” in our sense, and “normal” or “(T4)” means “both (T1) and (T4)” in our sense; in some other books, “regular” has the same meaning as our’s, while “(T3)” means “both (T1) and (T3)” in our sense, and likewise with the meanings of “normal” and “(T4)”.

First we give equivalent characterizations of these axioms.

**Proposition 1.1.** Let \((X, \mathcal{T})\) be a topological space.

(1) \((X, \mathcal{T})\) is (T1) if and only if

\[
\text{Any single point set } \{x\} \text{ is closed.}
\]

(2) \((X, \mathcal{T})\) is (T2) if and only if

\[
\text{The diagonal } \Delta = \{(x,x) | x \in X\} \text{ is closed in } X \times X.
\]

(3) \((X, \mathcal{T})\) is (T3) if and only if

\[
\forall x \in U \text{ open}, \exists V \text{ open such that } x \in V \subset V \subset U.
\]

(4) \((X, \mathcal{T})\) is (T4) if and only if

\[
\forall \text{ closed } A \subset U \text{ open}, \exists V \text{ open such that } A \subset V \subset V \subset U.
\]

**Proof.**

(1) \((\Rightarrow)\) For \(\forall y \neq x, \exists U_y \in \mathcal{T}\) such that \(x \notin U_y\). So

\[
\{x\}^c = \bigcup_{y \neq x} U_y
\]

is open, i.e. \(\{x\}\) is closed.

\((\Leftarrow)\) For \(\forall x \neq y\), take

\[
U = \{y\}^c \text{ and } V = \{x\}^c.
\]

Then \(x \notin V, y \notin U\) and \(x \in U, y \in V\).
(2) \( \Rightarrow \) For \( \forall x \neq y \), \((T2)\) implies \( \exists \) open set \( U_x \times V_y \) in \( X \times X \) such that \[(x, y) \in U_x \times V_y \quad \text{and} \quad \Delta \cap (U_x \times V_y) = \emptyset.\]

So \( \Delta^c \) is open, i.e. \( \Delta \) is closed.

\( \Leftarrow \) For \( \forall x \neq y \), i.e. \( (x, y) \in \Delta^c \), \( \exists \) open sets \( U \ni x \), \( V \ni y \) such that \[(x, y) \in U \times V \subset \Delta^c.\]

It follows \( U \cap V = \emptyset \), since if \( z \in U \cap V \), then \[(z, z) \in (U \times V) \cap \Delta = \emptyset.\]

(3) \( \Rightarrow \) Suppose \( x \in U \) open, i.e. \( x \notin U^c \) closed, then there exists \( V_1, V_2 \in \mathcal{T} \) such that
\[V_1 \cap V_2 = \emptyset, \ x \in V_1, \text{ and } U^c \subset V_2.\]

So \( x \in V_1 \subset \overline{V_1} \subset V_2^c \subset U.\)

\( \Leftarrow \) Suppose \( x \notin A \) closed, i.e. \( x \in A^c \) open, then there exists \( V \in \mathcal{T} \) such that \( x \in V \subset \overline{V} \subset A^c.\) It follows
\[V \cap \overline{V}^c = \emptyset, \ x \in V, \text{ and } A \subset \overline{V}^c.\]

(4) \( \Rightarrow \) Suppose \( A \subset U \) open, then \( A \cap U^c = \emptyset \). So there exists \( V_1, V_2 \in \mathcal{T} \) s.t.
\[V_1 \cap V_2 = \emptyset, \ A \subset V_1, \text{ and } U^c \subset V_2.\]

So \( A \subset V_1 \subset \overline{V_1} \subset V_2^c \subset U.\)

\( \Leftarrow \) Suppose \( A, B \) are closed and \( A \cap B = \emptyset \). Then \( A \subset B^c \) open. So there exists \( V \in \mathcal{T} \) such that \( A \subset V \subset \overline{V} \subset B^c.\) It follows that
\[V \cap \overline{V}^c = \emptyset, \ A \subset V \text{ and } B \subset \overline{V}^c.\]

\[ \square \]

We can also study the relations between these axioms. Obviously we have

- \( (T2) \implies (T1) \)
- \( (T1)+(T3) \implies (T2) \)
- \( (T1)+(T4) \implies (T2) \)
- \( (T1)+(T4) \implies (T3) \)

Note: We have

- \( (T1) \not\implies (T2) \)
- \( (T1) \not\implies (T3) \)
- \( (T1) \not\implies (T4) \)

Counterexample: \((\mathbb{R}, \mathcal{T}_{cofinite})\)

- \( (T4) \not\implies (T3) \)
- \( (T4) \not\implies (T2) \)
- \( (T4) \not\implies (T1) \)

Counterexample: \((\mathbb{R}, \mathcal{T}), \) where \( \mathcal{T} = \{(-\infty, a)|a \in \mathbb{R}\}.\)
[It is \( (T4) \) because there exists no disjoint closed sets at all!]
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- \((T3) \Rightarrow (T2)\), \((T3) \Rightarrow (T1)\).
  Counterexample: \((\mathbb{R}, T)\), where \(T\) is generated by
  \(B = \{[2n, 2n + 2] | n \in \mathbb{Z}\}\).
  In this topology, closed subsets and open subsets are the same.

- \((T2) \Rightarrow (T4)\), \((T2) \Rightarrow (T3)\).
  Counterexample: \((\mathbb{R}, T)\) where \(T\) is generated by
  \(S = \{(a, b) | a, b \in \mathbb{Q}\} \cup \{\mathbb{Q}\}\).
  \(\mathbb{Q}^c\) is closed but it can’t be separated from \(\{0\}\).

- \((T3) \Rightarrow (T4)\).
  Counterexample: The Sorgenfrey plane \((\mathbb{R}, T_{\text{sorgenfrey}}) \times (\mathbb{R}, T_{\text{sorgenfrey}})\).
  [Section 31 (Page 152) on Munkres’ book.]

It turns out that the compactness “enhances” the separation axioms.

**Proposition 1.2.** For topological spaces, we have

- \(\text{Compact + (T2) } \implies (T3)\).
- \(\text{Compact + (T3) } \implies (T4)\).

**Proof.**

- \([\text{Compact + (T2) } \implies (T3)]\)
  Let \(x \in X, A \subset X\) be closed (and thus compact), and \(x \notin A\). Then for any \(y \in A\), there exists open sets \(U_{x,y} \ni x, V_y \ni y\) such that
    \[U_{x,y} \cap V_y = \emptyset.\]
  By compactness of \(A\), \(\exists V_{y_1}, \ldots, V_{y_n}\) cover \(A\). It follows that
    \[U := U_{x,y_1} \cap \cdots \cap U_{x,y_n}\]
  is an open neighbourhood of \(x\),
    \[V := V_{y_1} \cup \cdots \cup V_{y_n}\]
  is an open neighbourhood of \(A\), and \(U \cap V = \emptyset\).

- \([\text{Compact + (T3) } \implies (T4)]\)
  Let \(A, B\) be disjoint closed subsets. Then for any \(x \in A\), there exists open sets \(U_x \ni x, V_x \supset B\) such that
    \[U_x \cap V_x = \emptyset.\]
  By compactness, \(\exists U_{x_1}, \ldots, U_{x_n}\) cover \(A\). It follows that
    \[U := U_{x_1} \cup \cdots \cup U_{x_n}\]
is an open neighbourhood of $A$, 

$$V := V_{x_1} \cap \cdots \cap V_{x_n}$$

is an open neighbourhood of $B$, and $U \cap V = \emptyset$.

Countability also “enhances” the separation axiom.

**Proposition 1.3.** We have

$$(A2) + (T3) \implies (T4)$$

**Proof.** In PSet 4-1-1(a) we have seen that any second countable space is Lindelöf. So the proposition follows from the following stronger result,

$Lindelöf + (T3) \implies (T4)$

To prove this, we let $A, B$ be disjoint closed subsets in $X$. Note that $A, B$ are also Lindelöf. Since $X$ is (T3), for any $x \in A$, one can find open set $V_x$ such that

$$x \in V_x \subset \overline{V_x} \subset B^c.$$ 

Since these $V_x$’s cover $A$ which is Lindeöf, we can choose a countable sub-covering $V_1, V_2, \cdots$ that covers $A$. Similarly one can find countably many open sets $U_1, U_2, \cdots$ that cover $B$ and

$$U_i \subset \overline{U_i} \subset A^c.$$ 

Now let

$$G_n := V_n \setminus (\bigcup_{i=1}^{n} U_i) \quad \text{and} \quad H_n := U_n \setminus (\bigcup_{i=1}^{n} V_i).$$

Then

$$A \subset \left( \bigcup_{n=1}^{\infty} V_n \right) \cap \left( \bigcap_{i=1}^{\infty} \overline{U_i}^c \right) \subset \bigcup_{n=1}^{\infty} \left( V_n \cap \bigcap_{i=1}^{n} \overline{U_i}^c \right) = \bigcup_{n=1}^{\infty} G_n$$

and similarly we have

$$B \subset \bigcup_{m=1}^{\infty} H_m.$$ 

Finally,

$$\left( \bigcup_{n=1}^{\infty} G_n \right) \cap \left( \bigcup_{m=1}^{\infty} H_m \right) = \emptyset$$

since $G_n \cap H_m = \emptyset$ holds for all $n, m$. [Check this!]
2. URYSOHN LEMMA

In this section we will prove Urysohn’s lemma. Urysohn’s lemma is a fundamentally important tool in topology using which one can construct continuous functions with certain properties. For example, we have seen last time how to use Urysohn’s lemma to prove Urysohn metrization theorem. Other important applications of Urysohn’s lemma include Tietze extension theorem, and embedding manifolds to Euclidean spaces. For the case of metric spaces, we have seen the proof which is very simple, because we already have a very nice continuous function – the distance function. However, the proof to the general version for normal spaces is very non-trivial.

**Theorem 2.1** (Urysohn Lemma). A topological space \((X, \mathcal{T})\) is normal if and only if for any pair of disjoint closed subsets \(A, B \subset X\), there exists a continuous function \(f : X \to [0, 1]\) such that

\[
A \subset f^{-1}(0) \quad \text{and} \quad B \subset f^{-1}(1).
\]

**Proof.**

\((\Leftarrow)\) This is the easy part: If

\[
A \subset f^{-1}(0), B \subset f^{-1}(1)
\]

for some continuous function \(f : X \to [0, 1]\), then \(f^{-1}([0, \frac{1}{3}])\) and \(f^{-1}((\frac{2}{3}, 1])\) are disjoint open neighbourhoods of \(A\) and \(B\). So \((X, \mathcal{T})\) is normal.

\((\Rightarrow)\) (This is the hard part! How do we define a continuous function on a very general topological space? Idea: “isoheight lines”.)

**Step 1:** Suppose we have a closed set \(A\) inside an open set \(U\). We denote \(A = A_0, U = U_1\). Since \(X\) is normal, we can find open set \(U_1\) and closed set \(A_1\) (which can be taken to be \(U_1\) if you want), such that

\[
A_0 \subset U_\frac{1}{2} \subset A_\frac{1}{2} \subset U_1.
\]

Repeat the same procedure twice more, we get

\[
A_0 \subset U_\frac{1}{4} \subset A_\frac{1}{4} \subset U_\frac{1}{2} \subset A_\frac{1}{2} \subset U_\frac{3}{4} \subset A_\frac{3}{4} \subset U_1.
\]

By induction, we can construct, for each dyadic rational number

\[
r \in D := \left\{ \frac{m}{2^n} \mid n, m \in \mathbb{N}, 1 \leq m \leq 2^n \right\}
\]

open sets \(U_r\) closed sets \(A_r\), such that

\[
1. U_r \subset A_r, \forall r \in D.
2. A_r \subset U_{r'}, \forall r < r' \in D.
\]

**Step 2:** Now we define

\[
f(x) = \inf\{r : x \in U_r\} = \inf\{r : x \in A_r\},
\]
where we “define” $\inf \emptyset = 1$. Clearly we have 
\[ A \subset f^{-1}(0) \quad \text{and} \quad B = U^c \subset f^{-1}(1). \]
It remains to prove $f$ is continuous. Since 
\[ \{(-\infty, \alpha) | \alpha \in D\} \cup \{(\alpha, +\infty) | \alpha \in D\} \]
is a sub-base for the topology of $[0, 1]$, it remains to prove that $f^{-1}((-\infty, \alpha))$ and $f^{-1}((\alpha, +\infty))$ are open for $\forall \alpha \in D$, which follows from the facts

\[ f^{-1}((-\infty, \alpha)) = \bigcup_{r<\alpha} U_r \quad \text{and} \quad f^{-1}((\alpha, +\infty)) = \bigcup_{r>\alpha} A^c_r. \]

□

Note that the conclusion of Urysohn’s lemma is 
\[ A \subset f^{-1}(0), \quad B \subset f^{-1}(1). \]

A natural question is: Under the same assumptions, can we construct continuous function $f$ such that 
\[ A = f^{-1}(0), \quad B = f^{-1}(1)? \]
To answer this question, let’s first consider the following more fundamental problem:

Question: What is the necessary condition for a set $A \subset (X, T)$ so that there exists a continuous function $f : X \to \mathbb{R}$ such that $f^{-1}(0) = A$?

Of course we need $A$ to be a closed set. But that is NOT enough: since 
\[ \{0\} = \bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}), \]
we must have 
\[ f^{-1}(0) = \bigcap_{n=1}^{\infty} f^{-1}\left((-\frac{1}{n}, \frac{1}{n})\right). \]
In other words,

\[ f^{-1}(0) \text{ should be the intersection of countably many open sets in } X. \]

**Definition 2.2.**

1. We say $A \subset (X, \mathcal{T})$ is a $G_\delta$-set if it is a countable intersection of open sets;
2. We say $A \subset (X, \mathcal{T})$ is an $F_\sigma$-set if it is a countable union of closed sets.

**Example.**

1. $\mathbb{Q} \subset \mathbb{R}$ is a $F_\sigma$-set, $\mathbb{R} \setminus \mathbb{Q} \subset \mathbb{R}$ is a $G_\delta$-set.
2. Any closed subset $F$ in any metric space $(X, d)$ is a $G_\delta$-set since 
\[ F = \bigcap_{n=1}^{\infty} \left\{ x \mid d(x, F) < \frac{1}{n} \right\}. \]
Consider the space $X = \{0, 1\}^\mathbb{R}$, equipped with the product topology. Then $X$ is compact (by Tychonoff), and $X$ is Hausdorff since it is the product of Hausdorff spaces (see today’s PSet). As a consequence, $X$ is a (T4) space, and each single point set $\{a\}$ is closed. However, $\{a\}$ is NOT $G_\delta$-set, since any non-empty $G_\delta$-set must contain uncountably many points: Each open set $U$ has only finitely many “non-$\{0, 1\}$” positions (think about a sub-base), which implies that each $G_\delta$-set has only countably many “non-$\{0, 1\}$” positions, and thus contains uncountably many elements.

Now we are ready to answer the question we mentioned above.

**Theorem 2.3** (A variant of Urysohn’s lemma). Let $(X, \mathcal{T})$ be a normal space, and $A, B \subset X$. Then there exists a continuous function $f : X \to [0, 1]$ such that

$$f^{-1}(0) = A, \quad f^{-1}(1) = B$$

if and only if $A, B$ are disjoint closed $G_\delta$-sets in $X$.

**Proof.** Obviously if such an $f$ exists, then $A, B$ must be disjoint, closed $G_\delta$-sets.

Conversely let $A, B$ be disjoint, closed $G_\delta$-sets.

**Claim:** There exists a continuous function $f_1 : X \to [0, 1]$ such that $f_1^{-1}(0) = A$.

**Proof.** Since $A$ is a $G_\delta$-set, there exists open sets $U_n$ in $X$ such that $A = \bigcap_{n=1}^{\infty} U_n$. By Urysohn’s lemma, there exists continuous functions $g_n : X \to [0, 1]$ such that

$$A \subset g_n^{-1}(0), \quad U_n^c \subset g_n^{-1}(1).$$

Now we define

$$f_1(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} g_n(x).$$

Then $f_1$ is continuous (since the continuous functions $\sum_{n=1}^{m} \frac{1}{2^n} g_n(x)$ converges to $f_1(x)$ uniformly.). Moreover, we have

$$f_1^{-1}(0) = A$$

since $A \subset f_1^{-1}(0)$, and for any $x \notin A$, there exists $n$ such that $x \in U_n^c$, i.e. $g_n(x) = 1$, which implies $f(x) \neq 0$. □

As a consequence, we can find continuous functions $f_i : X \to [0, 1], i = 1, 2$ such that

$$f_1^{-1}(0) = A, f_2^{-1}(0) = B.$$

Since $A \cap B = \emptyset$, we must have $f_1 + f_2 > 0$ on $X$. Finally we define:

$$f(x) = \frac{f_1(x)}{f_1(x) + f_2(x)}, \quad \forall x \in X.$$

Then $f : X \to [0, 1]$ is continuous, and $f^{-1}(0) = A, f^{-1}(1) = B$. □