Topology (H) Lecture 10
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TIETZE EXTENSION THEOREM

Last time we learned:

- Separation axioms: (T1), (T2), (T3), (T4)
  - The only "(Ti) $\Longrightarrow$ (Tj)" ($1 \le i \ne j \le 4$) is $\boxed{\text{(T2)} \Longrightarrow \text{(T1)}}$
  - $\boxed{\text{(T1) + (T4)} \Longrightarrow \text{(T3)}}$, $\boxed{\text{(T1) + (T3)} \Longrightarrow \text{(T2)}}$
  - $\boxed{\text{compact + (T3)} \Longrightarrow \text{(T4)}}$, $\boxed{\text{compact + (T2)} \Longrightarrow \text{(T3)}}$
  - $\boxed{\text{Lindelöf + (T3)} \Longrightarrow \text{(T4)}}$
- Urysohn's lemma: $(X, \mathscr{T})$ is (T4) if and only if disjoint closed subsets can be separated by a continuous map.
  - A, B disjoint closed sets in (T4) space $\Longrightarrow \exists$ continuous $f : X \to [0,1]$ such that $f(A) = 0, f(B) = 1$.
  - In a (T4) space, $A = f^{-1}(0)$ for some continuous function $f : X \to [0,1]$ $\Longleftrightarrow$ A is a closed G_δ set.

1. Tietze Extension Theorem

Although the function we get in Urysohn's lemma looks too special, they can be used as building blocks to construct more complicated continuous functions with certain properties, as we have seen in the proof of Urysohn's metrization theorem and in the proof of the variant of Urysohn's lemma. In this section we will give another application of Urysohn's lemma, Tietze extension theorem,[1] which can be viewed as a generalization of Urysohn's lemma (although they are in fact equivalent), and thus is directly applicable to more situations.

We start with a trivial definition.

Definition 1.1. Let $A \subset X$ be a subset. We say a map $\tilde{f} : X \to Y$ is an *extension* of a map $f : A \to Y$ if $\tilde{f} = f$ on A.

In analysis, it is always important to extend a given function from a smaller domain to a larger domain, while keeping some properties, e.g. continuity (or smoothness),

[1] According to wikipedia, the theorem was first proved by Brouwer and Lebesgue for the special case of the theorem when X is $\mathbb{R}^n$, and then was extended by Tietze to all metric spaces. The current version for normal space was proved by Urysohn.

boundedness. As a result, Tietze extension theorem is one of the most useful theorems in topology.

Theorem 1.2 (Tietze Extension Theorem). *A topological space $(X, \mathscr{T})$ is normal if and only if for any closed set $A \subset X$ and given any continuous function $f : A \to [0,1]$, there exists a continuous function $\tilde{f} : X \to [0,1]$ which is an extension of f.*

Proof.

($\Leftarrow$) Let A, B be disjoint closed sets in X. Then $A \cup B$ is closed in X, and

$$f : A \cup B \to [0,1], \quad f(x) = \begin{cases} 0, & x \in A \\ 1, & x \in B \end{cases}$$

is a continuous function on $A \cup B$. By assumption, f can be extended to a continuous function $\tilde{f} : X \to [0,1]$ such that $\tilde{f} = f$ on $A \cup B$. So by Urysohn's lemma, X is (T4).

($\Rightarrow$) **The idea**

Consider the "restriction map"

$$R : \mathcal{C}(X,[0,1]) \to \mathcal{C}(A,[0,1]), \quad g \mapsto g|_A,$$

where $\mathcal{C}(X,[0,1])$ means the space of all continuous maps $f : X \to [0,1]$. Then we want to prove that R is surjective, i.e. we want to solve the equation

$$Rg = f.$$

We will apply a standard trick in analysis:

① First find an approximate solution.
② Then iteratively find better and better approximations.
③ Finally prove the sequence of approximate solutions converges to a true solution.

Now we realize the idea. For simplicity, we replace $[0,1]$ by $[-1,1]$.

Step 1 [Construction an approximate solution]

First we approximate the function f by

$$f_1 : A \to \mathbb{R}, \qquad f_1(x) = \begin{cases} 1/3, & \text{if } f(x) \geqslant 1/3, \\ f(x), & \text{if } |f(x)| \leqslant 1/3, \\ \text{-}1/3, & \text{if } f(x) \leqslant -1/3. \end{cases}$$

By construction we have

$$|f(x) - f_1(x)| \leqslant \frac{2}{3}, \quad \forall x \in A.$$

Then we use Urysohn's lemma to find a continuous function $g : X \to \mathbb{R}$ s.t.

$$Rg \approx f_1.$$

There is a very obvious candidate for such a function g: since

$$A_1 := \left\{x \in A \middle| f(x) \leqslant \frac{1}{3}\right\} \quad \text{and} \quad B_1 := \left\{x \in A \middle| f(x) \leqslant -\frac{1}{3}\right\}$$

are both closed in X, there exists a continuous $g : X \to [-\frac{1}{3}, \frac{1}{3}]$ s.t.

$$g(x) = \frac{1}{3} \text{ on } A_1, \qquad g(x) = -\frac{1}{3} \text{ on } B_1.$$

It's easy to see that $g(x)$ also satisfies

$$|f(x) - Rg(x)| \leqslant \frac{2}{3}, \ \forall x \in A.$$

Step 2 [Do iteration]

Write $f = f_1$. According to Step 1, we have obtained a continuous function $g_1 : X \to [-\frac{1}{3}, \frac{1}{3}]$ s.t.

$$|f_1(x) - Rg_1(x)| \leqslant \frac{2}{3}, \ \forall x \in A.$$

Repeat Step 1 with f replaced by $f_2 = f_1 - Rg_1$, we can construct a continuous function $g_2 : X \to [-\frac{1}{3} \cdot \frac{2}{3}, \frac{1}{3} \cdot \frac{2}{3}]$ s.t.

$$|f_2(x) - Rg_2(x)| \leqslant (\frac{2}{3})^2, \ \forall x \in A.$$

Iteratively we can find a sequence of continuous functions

$$g_n : X \to \left[-\frac{1}{3}(\frac{2}{3})^{n-1}, \frac{1}{3}(\frac{2}{3})^{n-1}\right]$$

s.t. if we denote $f_{n+1} = f_n - Rg_n$, then

$$|f_n(x) - Rg_n(x)| \leqslant (\frac{2}{3})^n, \ \forall x \in A.$$

Step 3 [Converges to a solution]

Let

$$\widetilde{f}(x) = \sum_{n=1}^{\infty} g_n(x).$$

Since each g_n is continuous on X, and

$$|g_n(x)| \leqslant \frac{1}{3}(\frac{2}{3})^{n-1},$$

we see the series converges uniformly and thus $\widetilde{f}$ is continuous on X, and

$$|\widetilde{f}(x)| \leqslant 1, \quad \forall x \in X.$$

Finally for $\forall x \in A$, we have

$$\begin{aligned}\left|f(x) - \sum_{n=1}^{N} g_n(x)\right| &= |f_1 - g_1 - \cdots - g_N| \\ &= |f_2 - g_2 - \cdots - g_N| \\ &= \cdots \\ &= |f_N - g_N| \\ &\leqslant (\frac{2}{3})^N,\end{aligned}$$

So $f(x) = \widetilde{f}(x)$ for $x \in A$.

□

2. Various generalizations

Obviously in the statement of Tietze extension theorem, we can replace the target space $[0, 1]$ by any closed interval $[a, b]$: We only need to compose the functions we get with the linear transform

$$t \mapsto a + t(b - a)$$

and its inverse transform.

A not-that-obvious extension is: we can replace $[0, 1]$ by $\mathbb{R}$.

Theorem 2.1 (Tietze extension theorem for unbounded functions)**.** *Suppose X is normal and $A \subset X$ is closed. Then any continuous function $f : A \to \mathbb{R}$ can be extended to a continuous function $\widetilde{f} : X \to \mathbb{R}$.*

Proof. Composing f with the function $\arctan(x)$, we get a continuous function

$$f_1 := \arctan \circ f : A \to (-\frac{\pi}{2}, \frac{\pi}{2}).$$

By Tietze extension theorem, we can extend f_1 to a continuous function

$$\widetilde{f_1} : X \to [-\frac{\pi}{2}, \frac{\pi}{2}].$$

Let

$$B = \widetilde{f_1}^{-1}(\pm\frac{\pi}{2}).$$

Then B is closed in X, and $B \cap A = \emptyset$. By Urysohn's lemma, there exists a continuous function $g : X \to [0, 1]$ s.t.

$$g(A) = 1 \quad \text{and} \quad g(B) = 0.$$

Define

$$h(x) = \widetilde{f_1}(x)g(x).$$

Then h is a continuous function mapping X into $(-\frac{\pi}{2}, \frac{\pi}{2})$. Finally we let

$$\widetilde{f}(x) = \tan h(x).$$

Then $\widetilde{f} : X \to \mathbb{R}$ is continuous, and

$$\widetilde{f}(x) = \tan h(x) = \tan \widetilde{f_1}(x) = \tan f_1(x) = x, \ \forall x \in A.$$

□

Remark. One can also extend continuous vector-valued functions

$$f : A \to [0,1]^n, \quad f : A \to \mathbb{R}^n, \quad \text{or} \quad f : A \to [0,1]^S$$

to continuous vector-valued functions on X, i.e. to

$$\widetilde{f} : X \to [0,1]^n, \quad \widetilde{f} : X \to \mathbb{R}^n, \quad \text{or} \quad \widetilde{f} : X \to [0,1]^S,$$

where S is an arbitrary set. To do so, one only need to extend each component of f separately. [Note: A map to the product space is continuous if and only if all its components are continuous.]

On the other hand, for a topological space Y, in general one can't expect to extend any continuous function $f : A \to Y$ to a continuous function $\widetilde{f} : X \to Y$.

- To extend a function $f : \{0,1\} \to Y$ to a continuous map
$$\widetilde{f} : [0,1] \to Y,$$
a necessary condition is: $f(0)$ and $f(1)$ should lie in the same "path component" of Y.
- To extend a continuous map $f : S^1 \to Y$ to a continuous map $\widetilde{f} : D \to Y$, where D is the unit disc in the plane, one need require the image $f(S^1)$ to be "contractible" in Y. In particular, we will see that the identity map
$$f : S^1 \to S^1, x \mapsto x$$
can not be extended to a continuous map $\widetilde{f} : D \to S^1$. [Brouwer fixed point theorem]

We will study these connectivity phenomena in the second half of this course.

Remark. One may pose extra assumptions on the extension. For example, one can extend smooth functions to smooth functions, which is known as Whitney extension theorem. One can also require the extension to preserve other properties like Lipschitz/Hölder continuity (for metric space), or boundedness (See PSet).

To apply Urysohn'e lemma or Tietze extension theorem, one need to assume that the source space is normal. One class of normal spaces that appears in many applications are those topological spaces that are both compact and Hausdorff. However, in some other applications, compactness is too strong, and we only have "local compactness" and Hausdorff. Recall [PSet 3-2-2(c)]:

Definition 2.2. A topological space $(X, \mathscr{T})$ is called *locally compact* if for any $x \in X$, there exists a compact set K_x and an open set U_x such that

$$x \in U_x \subset K_x.$$

Notation. A locally compact Hausdorff space will be abbreviated as *LCH.*

Example.

- Any compact Hausdorff space is LCH.
- $\mathbb{R}^n$ is LCH.
- More generally, any locally Euclidian Hausdorff space is LCH.
  - We say a topological space is *locally Euclidian* if for any $x \in X$, there exists a neighborhood U of x which is homeomorphic to an open ball in Euclidian space.
- Let K be any compact Hausdorff space. Let p be any point such that

  $$X = K \setminus \{p\}$$

  is non-compact. Then

  Fact 1. X is a non-compact LCH.

  As a subspace of a Hausdorff space, X is Hausdorff. To see X is locally compact, for any $x \in X \subset K$, since K is Hausdorff we can find disjoint open sets U, V in K such that $x \in U$, $p \in V$. Now $K \setminus V$ is a closed subset in compact space K, and thus is compact in K. Since $K \setminus V \subset X$, it is also compact in X. By definition it is a compact neighborhood of x in X, since $U \subset K \setminus V$.

  Conversely,

  Fact 2. Any non-compact LCH arises this way:

  According to PSet3-1-3(b), any non-compact topological space admits a *one-point compactification* $X^* = X \cup \{\infty\}$ [2]. In the case X is a non-compact LCH, one can easily check that the X^* is a compact Hausdorff space. In particular, this shows that any LCH can be realized an an open subspace of some compact Hausdorff space.

  Remark. One can prove that the space $\mathbb{Q}_p$, i.e. the completion of $\mathbb{Q}$ under the p-adic metric, is locally compact, and thus LCH. As a result, analysis on LCH is very useful in p-adic analysis.

- $\mathbb{Q} \subset \mathbb{R}$ is NOT locally compact. [Why?]
- The Sorgenfrey line $(\mathbb{R}, \mathscr{T}_{Sorgenfrey})$ is NOT locally compact. [Why?]

[2] The one-point compactification is also known as the *Alexandrov compactification*, and sometimes denoted by αX.

LCHs appear widely in analysis. For LCHs, people also want to apply Urysohn's lemma or Tietze extension theorem to construct continuous functions with specific properties. Unfortunately, not all LCH's are normal, so that we can't extend all continuous functions defined on closed sets. [Note: normal is a necessary and sufficient condition in Urysohn'e lemma as well as in Tietze extension theorem.] However, we can prove that a weaker version holds for LCH, which is one of the most useful results in the analysis on LCH:

Theorem 2.3 (Urysohn's Lemma, LCH version)**.** *Let X be a LCH, and K, F be disjoint subsets in X with K compact and F closed. Then there exists a continuous function $f : X \to [0, 1]$ such that $f(K) = 1$ and $f(F) = 0$.*

Proof. The proof is based on the following separation property in LCH. [You can compare this with the equivalent characterization of (T4) that we proved last time.]

> **Lemma 2.4.** *Let X be a LCH, K be a compact set in X, and U be an open set in X such that $K \subset U$. Then there exists an open set V such that $\overline{V}$ is compact, and*
> $$K \subset V \subset \overline{V} \subset U.$$

We will leave the proof of the lemma as an exercise. Applying the lemma to the pair $K \subset F^c$, we can find an open set V with $\overline{V}$ compact, such that

$$K \subset V \subset \overline{V} \subset F^c.$$

Applying the lemma again, but now to $K \subset V$, we will find an open set U with $\overline{U}$ compact, such that

$$K \subset U \subset \overline{U} \subset V.$$

Since $\overline{V}$ is a compact Hausdorff space, it is normal. According to Urysohn's lemma, one can find a continuous function $g : \overline{V} \to [0, 1]$ such that

$$g(K) = 1, \quad g(\overline{V} \setminus U) = 0.$$

We can extend g to a continuous function $f : X \to [0, 1]$ by simply "extend by 0", i.e.

$$f(x) = \begin{cases} g(x), & x \in \overline{V}, \\ 0, & x \in X \setminus \overline{V}. \end{cases}$$

Then f is continuous on both V and $X \setminus \overline{U}$. It follows that f is continuous on X. Obviously $f(K) = 1$ and $f(F) = 0$. □

In a very similar way, one can prove the following LCH version of Tietze extension theorem. We will leave the proof as an exercise as well:

Theorem 2.5 (Tietze extension theorem, LCH version)**.** *Let X be a LCH, and K be a compact subset. Then any continuous function $f : K \to \mathbb{R}$ can be extended to a compactly supported continuous function $f : X \to \mathbb{R}$.*

3. Various applications

Tietze extension theorem has many applications. For example, in real analysis, Tietze extension theorem was used to produce a sequence of continuous functions that approximates (a.e.) a given measurable function. In what follows we give more applications of Tietze extension theorem.

Application 1: Yet another characterization of compactness for metric spaces.

We have given several different characterization of compactness for metric spaces. Here is another one:

Proposition 3.1. *A metric space (X, d) is compact if and only if any continuous function $f : X \to \mathbb{R}$ is bounded.*

Proof. If (X, d) is compact, then by the extreme value theorem, any continuous function $f : X \to \mathbb{R}$ is bounded.

To prove the converse, we argue by contradiction. Suppose (X, d) is non-compact, then there exists $A = \{x_1, x_2, \cdots\}$ such that $A' = \emptyset$. It follows that A is closed and each x_n is isolated in A. So the function

$$f : A \to \mathbb{R}, f(x_n) = n$$

is continuous on A. By Tietze extension theorem, there exists a continuous function $\widetilde{f} : X \to \mathbb{R}$ such that $\widetilde{f} = f$ on A. Obviously $\widetilde{f}$ is an unbounded continuous function on X, a contradiction. □

Application 2: From the Cantor set to space-filling curves

Our second application is concerned with the Cantor set C. Recall

$$C = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} (\frac{3k+1}{3^n}, \frac{3k+2}{3^n}).$$

As we mentioned at the beginning of Lecture 7, one way to understand C is via the ternary representation of real numbers, i.e. regard C as the image of the map

$$g : \{0,1\}^{\mathbb{N}} \to [0,1], \quad a = (a_1, a_2, \cdots) \mapsto \sum_{k=1}^{\infty} \frac{2}{3^k} a_k.$$

We have checked in PSet 3-2-1(a)) that the map g is a homeomorphism from $(\{0,1\}^{\mathbb{N}}, \mathscr{T}_{product})$ onto the Cantor set C.

On the other hand, the map

$$h : \{0,1\}^{\mathbb{N}} \to [0,1]^2, \quad a = (a_1, a_2, \cdots) \mapsto \left(\sum_{k=1}^{\infty} \frac{a_{2k-1}}{2^k}, \sum_{k=1}^{\infty} \frac{a_{2k}}{2^k} \right)$$

is continuous and surjective:

To check continuity, one only need to check continuity of each component, which can be done easily via sub-base. The surjectivity is just another way to say that each real number has a binary representation. [Note: h can't be injective, otherwise as a bijective continuous map from a compact space to a Hausdorff space, it will has to be a homeomorphism, which is absurd.]

As a consequence, we get a continuous surjective map

$$h \circ g^{-1} : C \to [0,1]^2.$$

Since C is closed in $[0,1]$, Tietze extension theorem indicates that there exists a continuous surjective map

$$f : [0,1] \to [0,1]^2.$$

Definition 3.2. Any continuous surjective map from $[0,1]$ to $[0,1]^2$ is called a *Peano Curve* or a *space-filling curve.*

Remarks.

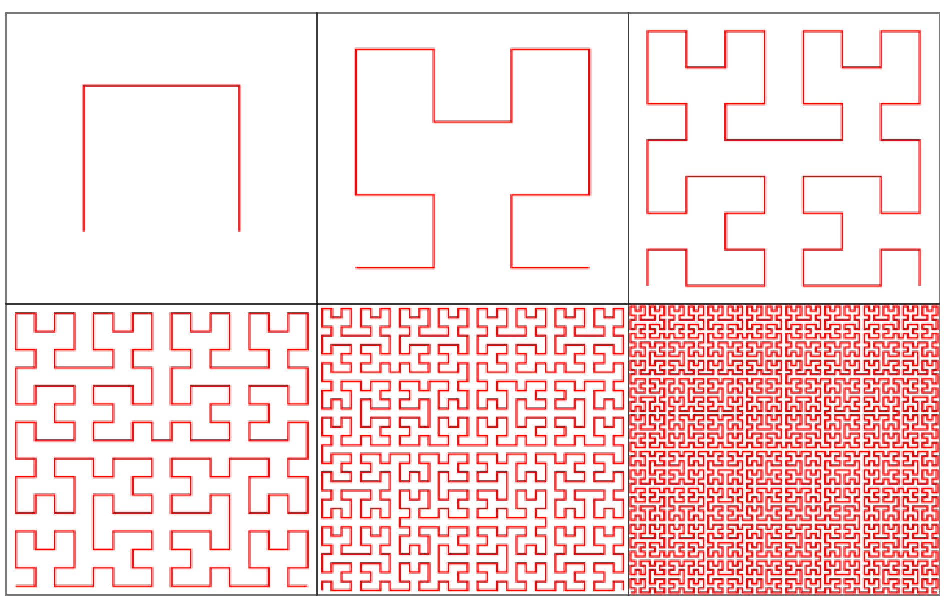
(1) By using a very similar argument, one can easily construct surjective continuous map

$$f : [0,1] \to [0,1]^n$$

or even surjective continuous map

$$f : [0,1] \to [0,1]^{\mathbb{N}}.$$

(2) Of course our argument is an "non-constructive" proof of the existence Peano curve. In literature there are also many "constructive proofs" which iteratively construct such a curve.



(3) The space-filling curves are not just theoretic monsters. They have many practical applications in real life. For example, it is used in storing multidimensional data into computer (which is arranged linearly), e.g. Google maps, so that when you move a little bit on the map, you only move a little bit in the memory, that is why we require continuity of the function.

Application 3: The Stone-Čech compactification of LCHs.

Let X be a non-compact LCH. We have seen how to compactify X to a compact Hausdorff space via the Alexandrov compactification (i.e. the-one point compactification). Geometrically, the Alexandrov compactification αX of X is a compactification that glue all "open ends" of X together, which is not as good as we want in many applications. Here is another widely used way to compactify X, via continuous functions.

As usual we will denote by $\mathcal{C}(X,[0,1])$ the space of all continuous functions $f : X \to [0,1]$. Consider the "huge product space"

$$Q = [0,1]^{\mathcal{C}(X,[0,1])},$$

equipped with the product topology. Since compactness and Hausdorff are both productive [PSet 4-2-2], Q is a compact Hausdorff space.

Theorem 3.3. *Let X be a LCH. Then the map θ defined by*

$$\theta : X \to Q, \quad x \mapsto (f(x))_{f\in\mathcal{C}(X,[0,1])}.$$

is a homeomorphism to its image $\theta(X)$.

Proof. For simplicity we denote

$$\beta X = \overline{\theta(X)}.$$

It is compact since it is a closed subset in the compact space Q. It is Hausdorff since it is a subspace of the Hausdorff space Q.

Before we prove the theorem, first note that any $f \in \mathcal{C}(X,[0,1])$ induces a map

$$\pi_f : Q \to [0,1]$$

is continuous since it is just the coordinate map. We denote

$$\tilde{f} = \pi_f|_{\beta X} : \beta X \to [0,1].$$

Then $\tilde{f}$ is continuous and satisfies $\tilde{f}\circ\theta = f$. Since βX is the closure of $\theta(X)$, such a continuous function $\tilde{f} : \beta X \to [0,1]$ is unique. Moreover, by our construction, if $\text{supp} f \subset K$ for some compact set K, then $\tilde{f} = 0$ on $\beta X \setminus \theta(K)$.

To prove $\theta : X \to \theta(X)$ is a homeomorphism, it is enough to prove f is continuous, bijective and open.

- θ is continuous since each component of θ is continuous.

- θ is injective (and thus bijective onto $\theta(X)$): For any $x \neq y$, by Urysohn lemma one can find continuous function $f \in \mathcal{C}(X,[0,1])$ such that $f(x) \neq f(y)$. Then the continuous function $\tilde{f}$ satisfies $\tilde{f}(\theta(x)) \neq \tilde{f}(\theta(y))$. So $\theta(x) \neq \theta(y)$.
- θ is open: For any open set U in X, and any point $y = \theta(x) \in \theta(U)$, by locally compactness one can find a compact set $K \subset U$ such that $x \in \mathrm{Int}(K) \subset K$. [To get such K, one can apply Lemma 2.4 to the pair $\{x\} \subset U$. The set $\overline{V}$ we get there can be taken as K here.] According to LCH version of Urysohn's lemma, there exists $f \in \mathcal{C}(X,[0,1])$ such that $f(x) = 1$ and $\mathrm{supp}(f) \subset K$. Consider the function $\tilde{f} : \beta X \to [0,1]$ as constructed above. Then $\tilde{f} = 0$ on $\beta X \setminus \theta(U)$. It follows that the open set
$$V = \tilde{f}^{-1}((0,+\infty)) \subset \beta X$$
is contained in $\theta(U)$. Clearly we have $y \in V$ since $f(x) = 1$.

□

The closure of the image,
$$\beta X := \overline{\theta(X)},$$
is a compactification of X. It is known as the *Stone-Čech compactification* of X.

In the proof, we also get the following important property of the Stone-Čech compactification.

Proposition 3.4. *Let X be a non-compact LCH. Then any bounded continuous function $f : X \to \mathbb{R}$ can be "extended" uniquely to a continuous function $\tilde{f} : \beta X \to \mathbb{R}$.*

Remark. So a space may have many different compactifications. We can view each compactification $\overline{X}$ as a pair $(\iota, \overline{X})$, where $\iota : X \to \overline{X}$ is a homeomorphism from X to its image $\iota(X)$. We say a compactification $(\iota, \overline{X})$ of X is *finer* than another compactification $(\iota', \overline{X}')$, (and say $(\iota', \overline{X}')$ is *coarser* than $(\iota, \overline{X})$), if three exists a continuous map $g : \overline{X}' \to \overline{X}$ such that $\iota = \pi \circ \iota'$. One can prove that for non-compact LCHs, the Alexandrov compactification αX is the coarsest compactification, while the Stone-Čech compactification is the finest compactification.

Application 4: Partition of unity.

Our last application is to "Partition of unity", which we will have more to say next time. Today we will only prove a simple version:

Theorem 3.5 ("Partition of unity"). *Let K_α be closed subsets in a normal space X such that*
$$\bigcup_\alpha K_\alpha = X.$$
Let U_α be open neighbourhoods of K_α's which is locally finite, i.e.

$$\boxed{\forall x \in X, \exists \text{ open } U_x \ni x \text{ s.t. } U_x \cap U_\alpha \neq \emptyset \text{ for at most finitely many } \alpha\text{'s.}}$$

Then there exist continuous functions $f_\alpha : X \to [0,1]$ *such that*

①$f_\alpha > 0$ *on* K_α.
②$f_\alpha = 0$ *on* U_α^c.
③$\sum_\alpha f_\alpha(x) = 1, \forall x \in X$.

Proof. By Tietze extension theorem (in fact, Urysohn's lemma), there exist continuous functions $g_\alpha : X \to [0,1]$ such that

$$g_\alpha = 1 \text{ on } K_\alpha, \qquad g_\alpha = 0 \text{ on } U_\alpha^c.$$

Define

$$g(x) = \sum_\alpha g_\alpha(x).$$

Then on open set U_x, g is a finite sum of continuous functions. So g is well-defined and is continuous on each U_x, and hence g is well-defined and is continuous on the whole of X. Moreover,

$$g(x) \geq 1, \quad \forall x$$

since $\bigcup_\alpha K_\alpha = X$. Now we set

$$f_\alpha(x) = \frac{g_\alpha(x)}{g(x)}.$$

It is easy to check that f_α's are what we need. □

Remark. Next time we will discuss topological conditions which guarantee the local finiteness assumption above, as well as applications of "partition of unity".