Topology (H) Lecture 12
Lecturer: Zuoqin Wang
Time: April 30, 2020

PARACOMPACTNESS AND PARTITION OF UNITY

Last time we learned:

- Tietze extension theorem: X is (T4) if and only if any continuous function defined on any closed subset A can be extended to a continuous function on X.
  - Also for unbounded/complex-valued/vector-valued functions etc
  - For locally compact Hausdorff spaces (LCH),
    - Urysohn: separate compact set with closed set via continuous function,
    - Tietze: One can extend continuous functions defined on compact sets to compactly supported function defined on X.
- Many applications
  - Characterize compactness of (X, d), construct Peano curve.
  - Stone-Cěch compactification.
  - A preliminary version of "partition of unity": In a (T4) space, given a family of closed sets $\{F_\alpha\}$ which cover X, if there exists open neighborhoods U_α of F_α such that $\{U_\alpha\}$ is a locally finite family, then there exists a family of continuous functions $\rho_\alpha : X \to [0,1]$ satisfying $\rho_\alpha > 0$ on F_α and $\rho_\alpha = 0$ on U_α^c, such that
$$\sum \rho_\alpha(x) = 1, \quad \forall x \in X.$$

Such a family of non-negative continuous functions defined on a topological space with prescribed supports that add up to 1 everywhere is called a partition of unity. In most applications, we only have an open covering. Here is a formal definition of partition of unity subordinate to an arbitrary open covering:

Definition 0.1. Let $\{U_\alpha\}$ be an open covering of X. We say a family of functions $\{\rho_\alpha\}$ is a (continuous) *partition of unity* (P.O.U.) subordinate to $\{U_\alpha\}$ if

(1) Each $\rho_\alpha : X \to [0,1]$ is continuous. [Note: ρ_α is defined on the whole of X!]
(2) For each α, the support of ρ_α is contained in U_α, i.e.
$$\mathrm{supp}\rho_\alpha := \overline{\{x \in X : \rho_\alpha(x) \neq 0\}} \subset U_\alpha.$$
(3) The family $\{\mathrm{supp}\rho_\alpha\}$ of sets is *locally finite*, i.e. for any $x \in X$, there exists open set $U_x \ni x$ such that $\mathrm{supp}\rho_\alpha \cap U_x \neq \emptyset$ for finitely many ρ_α's.
(4) For any $x \in X$, $\sum_\alpha \rho_\alpha(x) = 1$.

Note that (1) and (3) guarantee the summation in (4) is a continuous function.

1. Paracompactness

We have already seen that for any compact space, any open covering has a finite sub-covering, and as a result, we can study global properties via local properties.

In general, one can't hope to "glue local information to global information" in a very general topological space. However, the *partition of unity* gives us a "weaker" way to "glue local data to get global data". For example, as we have seen at the end of last lecture, to get a continuous function, we can add infinitely many (possibly uncountably many) continuous functions: the only extra assumption we need is "local finiteness", i.e. $\underline{\text{near each point}}$ we are only adding finitely many continuous functions. Now a natural question is: in applications (nice examples), is it always possible to find such a locally finite open covering?

Example. Consider $X = \mathbb{R}^n$, and the open covering

$$\mathscr{U} = \{B(\mathbf{0}, r) | r > 0\}.$$

This is NOT a $\underline{\text{locally finite}}$ covering, since any point is contained in infinitely many balls of radius r, centered at $\mathbf{0}$. In particular, you can't add infinitely many continuous functions supported in these balls.

However, we notice that for any $a \in \mathbb{Z}^n$, by compactness we can pick finitely many points

$$x_{a,1}, \cdots, x_{a,m(a)}$$

such that

$$\overline{B(a, \sqrt{n})} \subset \bigcup_{i=1}^{m(a)} B(x_{a,i}, 1).$$

Also we notice that $\{B(a, \sqrt{n}) \mid a \in \mathbb{Z}^n\}$ is an open covering of X. It follows that

$$\widetilde{\mathscr{U}} = \{B(x_{a,i}, 1) \mid a \in \mathbb{Z}^n, 1 \leqslant i \leqslant m(a)\}$$

is an open covering of X. Moreover,

- each open set in $\widetilde{\mathscr{U}}$ is contained in an open set in $\mathscr{U}$.
- $\widetilde{\mathscr{U}}$ is a locally finite covering, since for any x, there are only finitely many integer points that are $(\sqrt{n} + 1)$-close to x!

So we can try to construct P.O.U. via $\widetilde{\mathscr{U}}$!

Now we give a couple definitions extending the previous example:

Definition 1.1. Let X be a topological space.

(1) We say a covering $\mathscr{A}$ of X is a *locally finite* covering if for any $x \in X$, there exists an neighbourhood U_x of x such that U_x intersect with only finitely many elements in $\mathscr{A}$.

(2) We say a covering $\widetilde{\mathscr{A}}$ of X is a *refinement* of another covering $\mathscr{A}$ of X if for any $\widetilde{A} \in \widetilde{\mathscr{A}}$, there exists $A \in \mathscr{A}$ such that $\widetilde{A} \subset A$.

Definition 1.2. We say a topological space $(X, \mathscr{T})$ is *paracompact* [1] if any open covering of X admits a refinement which is locally finite.

Example.

1. Any compact space is paracompact.
2. Any discrete space $(X, \mathscr{T}_{discrete})$ is paracompact.
3. Any closed subset of a paracompact space is still paracompact.

    Suppose X is paracompact and $A \subset X$ is closed. Let $\mathscr{U}$ be any open covering of A. Let $\mathscr{U}_1 = \mathscr{U} \cup \{A^c\}$. Then it is an open covering of X. By definition, there exists a locally finite refinement $\widetilde{\mathscr{U}_1}$ of $\mathscr{U}_1$. Let

    $$\widetilde{\mathscr{U}} = \{U \in \widetilde{\mathscr{U}_1} \mid U \not\subset A^c\}.$$

    It is easy to see $\widetilde{\mathscr{U}}$ is an open covering of A which is a locally finite refinement of $\mathscr{U}$.

   [However: an arbitrary subset of a paracompact space could fail to be paracompact.]
4. $\mathbb{R}^n$ is paracompact:

    We just showed a special open covering of $\mathbb{R}^n$ admits a locally finite refinement. Now we prove this for every open covering. So we let $\mathscr{U}$ be any open covering of $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, there exits $0 < r_x \leqslant 1$ and $U \in \mathscr{U}$ such that $B(x, r_x) \subset U$. Let

    $$\mathscr{U}_1 = \{B(x, r_x) | x \in \mathbb{R}^n\}.$$

    Then $\mathscr{U}_1$ is a refinement of $\mathscr{U}$. Now proceed as before: Any closed ball of the form

    $$\overline{B(a, \sqrt{n})}, \qquad a \in \mathbb{Z}^n$$

    can be covered by finitely many open balls in $\mathscr{U}_1$. Let $\widetilde{\mathscr{U}}$ be the collection of these balls. Then $\widetilde{\mathscr{U}}$ is again an open covering of $\mathbb{R}^n$, is a refinement of $\mathscr{U}$, and is locally finite.

If you stare at the proof of paracompactness for $\mathbb{R}^n$ for a while, you may have seen that the crucial fact we used to construct locally finite open covering are:

- any point x has a compact neighbourhood $\overline{B(x, \sqrt{n})}$. In other words, we are using the *local compactness*!
- the whole space can be covered by countably many such compact balls, that is some kind of global countability [like (A2) or Lindelöf].

---

[1] The conception of paracompactness was first introduced by Dieudonné in 1944.

A natural question is: does local compactness plus countability like (A2) imply paracompactness? Unfortunately the answer is no in general:

Example. Consider $X = \mathbb{R}$ with $\mathscr{T} = \{(-\infty, a) | a \in \mathbb{R}\}$. Then $(X, \mathcal{T})$ is locally compact since any set of the form $(-\infty, x]$ is compact, it is (A2) since

$$\mathscr{B} = \{(-\infty, r) \mid r \in \mathbb{Q}\}$$

is a countable base, but it is NOT paracompact since the open covering

$$\mathscr{U} = \{(-\infty, n) \mid n \in \mathbb{Z}\}$$

has no locally finite refinement.

What is missing in this example? This topology is bad because open sets are too large to separate points (so that there is no local finiteness), i.e. it is NOT Hausdorff!

It turns out that local compactness, countability and Hausdorff together implies paracompactness:

Proposition 1.3. *We have*

$$\boxed{\textit{Lindelöf + locally compact + (T2)} \Longrightarrow \textit{paracompact.}}$$

Since (A2) implies Lindelöf, we get as a consequence

$$\boxed{\textit{(A2) + locally compact + (T2)} \Longrightarrow \textit{paracompact.}}$$

Proof. Let X be Lindelöf, locally compact and Hausdorff.

Step 1. Prove X is (T3).

Suppose $x \in U$. By local compactness, we can find a compact set K such that $x \in \mathring{K}$. Since X is (T2), K must be closed and (T2). Since K is compact and (T2), it is (T3). Now let $W = U \cap \mathring{K}$. Then W is an open neighbourhood of x in K. So there exists an open neighbourhood V of x in K such that

$$x \in V \subset \mathrm{Cl}_K(V) \subset W,$$

where we used the notation $\mathrm{Cl}_K(V)$ to represent "the closure of V inside the topological space K", so as to distinguish with $\overline{V}$, the closure of V in X. However, we claim that V is in fact open in X, and $\mathrm{Cl}_K(V)$ coincides with $\overline{V}$:

- Since V is open in K, and $V \subset W$ which is open in X, V has to be open in X.
- Since K is closed in X, and $V \subset K$, we see $\overline{V} \subset K$. It follows $\overline{V} = \mathrm{Cl}_K(V)$.

So we does get an open set V in X such that

$$x \in V \subset \overline{V} \subset W \subset U,$$

which implies X is (T3).

Step 2. Prove X is paracompact.

Let $\mathscr{U} = \{U_\alpha\}$ be any open covering of X. For any $x \in X$, we choose $\alpha(x)$ such that $x \in U_{\alpha(x)}$. Since X is (T3), we can find open sets V_x and W_x such that

$$x \in V_x \subset \overline{V_x} \subset W_x \subset \overline{W_x} \subset U_{\alpha(x)}.$$

Now $\mathscr{V} = \{V_x\}$ is an open covering of X. Since X is Lindelöf, we can find a countable sub-covering

$$\{V_1, V_2, V_3, \cdots\} \subset \mathscr{V}.$$

We denote $R_1 = W_1$ and define iteratively

$$R_n = W_n \setminus (\overline{V_1} \cup \cdots \cup \overline{V_{n-1}}), \quad n > 1.$$

We claim that $\mathscr{R} = \{R_n\}$ is a locally finite refinement of $\mathscr{U}$:

- By construction, $\mathscr{R}$ is an open refinement of $\mathscr{U}$.
- $\mathscr{R}$ is a covering of X since for any x, if we let n be the least integer such that $x \in W_n$, then $x \notin \overline{V_1} \cup \cdots \cup \overline{V_{n-1}}$ since $\overline{V_i} \subset W_i$. So we must have $x \in R_n$.
- $\mathscr{R}$ is locally finite since for any $x \in X$, we can find n such that $x \in V_n$, and the open neighborhood V_n of x intersects with only finitely many elements in $\mathscr{R}$ since $V_n \cap R_m = \emptyset$ for all $m > n$.

So X is paracompact. □

If you are not lost in the proof, you may have found that in Step 1 we used only local compactness and (T2), and in Step 2 we used only Lindelöf and (T3). So in fact we proved the following two facts which are independent to each other and are interesting by themselves:

(1) $\boxed{\text{Locally compact} + \text{(T2)} \Longrightarrow \text{(T3)}}$,
(2) $\boxed{\text{Lindelöf} + \text{(T3)} \Longrightarrow \text{paracompact}}$.

Remark. Instead of (A2) or Lindelöf, some people use the condition "σ-compact" :

> **Definition 1.4.** X is called *σ-compact* if it can be written as a countable union of compact sets.

It is easy to see $\boxed{\sigma\text{-compact} \Longrightarrow \text{Lindelöf.}}$ In mathematics, σ often means "countable union" and δ means "countable intersection", e.g. G_δ-set, F_σ-set, σ-algebra etc. Here is another one:

> **Definition 1.5.** A family of sets $\mathscr{A}$ in X is called *σ-locally finite* if $\mathscr{A} = \cup_n \mathscr{A}_n$, where each $\mathscr{A}_n$ is a locally finite family.

Remark. Manifolds are paracompact since by definition they are topological spaces satisfying (T2) + (A2) + locally Euclidean ($\Longrightarrow$locally compact).

> **Definition 1.6.** A *topological manifold* is a topological space which is (T2), (A2) and locally Euclidean, i.e. for each $x \in X$ there exists a neighborhood U which is homeomorphic to $\mathbb{R}^n$.

Here is another big class of topological spaces which are paracompact:

Theorem 1.7 (Stone)**.** *Any metric space is paracompact.*

Proof. [2] Let $\mathscr{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ be any open covering of a metric space (X, d). By the well-ordering principle, there is a well-ordering $\preceq$ on Λ. Since any subset of Λ contains a minimal element, for any x there exists a unique $\alpha_x \in \Lambda$ such that

$$x \in U_\alpha \setminus \bigcup_{\beta \prec \alpha} U_\beta.$$

Now for each $\alpha \in \Lambda$, we iteratively define for $n \in \mathbb{N}$,

$$X_{\alpha,n} = \left\{ x \in X \;\middle|\; B(x, \frac{3}{2^n}) \subset U_\alpha \text{ and } x \notin \bigcup_{\beta \prec \alpha} U_\beta \cup \bigcup_{\beta \in \Lambda, k<n} V_{\beta,k} \right\}$$

and

$$V_{\alpha,n} = \bigcup_{x \in X_{\alpha,n}} B(x, \frac{1}{2^n}).$$

In what follows we prove $\mathscr{V} = \{V_{\alpha,n}\}$ is a locally finite open refinement of $\mathscr{U}$.

Obviously

- each $V_{\alpha,n}$ is open,
- $V_{\alpha,n} \subset U_\alpha$,
- for any x, let α be the minimal element as chosen above. Choose n such that $B(x, \frac{3}{2^n}) \subset U_\alpha$. Then either $x \in V_{\beta,k}$ for some $\beta \in \Lambda$ and $k < n$, or $x \in X_{\alpha,n} \subset V_{\alpha,n}$.

so $\mathscr{V}$ is an open refinement of $\mathscr{U}$ which cover X. It remains to prove local finiteness.

Now for each $x \in X$, we let α be the smallest index so that $x \in \cup_n V_{\alpha,n}$. Choose n, k such that $B(x, 2^{-k}) \subset V_{\alpha,n}$. Then one can show (via the triangle inequality)

(1) For any $l \geq n + k$, the ball $B(x, 2^{-n-k})$ intersects no $V_{\beta,l}$.
[Check this!]
(2) For any $l < n + k$, the ball $B(x, 2^{-n-k})$ intersects $V_{\beta,l}$ for at most one $\beta \in \Lambda$.
[Check this!]

It follows that any point x has an open neighborhood $B(x, 2^{-n-k})$ which intersects with no more than $n + k$ elements in $\mathscr{V}$. So $\mathscr{V}$ is locally finite. □

---

[2] The original proof given by Stone is involved. The elementary proof given here is due to Rudin.

2. Partition of unity

Although looks very complicated, paracompactness is very important, especially in developing analysis on manifolds (a class of topological spaces on which we can do analysis), because

> paracompactness allows us to construct partitions of unity,
> and you need partitions of unity to do anything!

As we have seen at the end of last lecture, we will need to apply Urysohn's lemma or Tietze extension theorem to construct P.O.U. So we need the background space to be normal. In general, a paracompact space could fail to be normal. However, we have: [Compare: compactness "enhances" separability]

Proposition 2.1.

1. *Paracompact + (T2) $\Longrightarrow$ (T3);*
2. *Paracompact + (T3) $\Longrightarrow$ (T4).*

Proof. We first prove a lemma.

Lemma 2.2. *If $\mathscr{A}$ is a locally finite collection of subsets*[3]*. Then*

$$\overline{\bigcup_{A\in\mathcal{A}} A} = \bigcup_{A\in\mathcal{A}} \overline{A}.$$

Proof. We have

$$\bigcup_{A\in\mathcal{A}} \overline{A} \subset \overline{\bigcup_{A\in\mathcal{A}} A}$$

since for each $A \in \mathcal{A}$ we have $\overline{A} \subset \overline{\bigcup_{A\in\mathcal{A}} A}$.

Conversely, for any $x \in \overline{\bigcup_{A\in\mathcal{A}} A}$, by the definition of local finiteness, there exists open neighborhood U_x of x such that U_x intersect with finitely A's in $\mathscr{A}$. Without loss of generality, denote these A's by $A_1, \cdots, A_m$. Then we must have

$$x \in \overline{A_1} \cup \cdots \cup \overline{A_m} \subset \bigcup_{A\in\mathcal{A}} \overline{A},$$

otherwise $U_x \setminus \bigcup_{i=1}^{m} \overline{A_i}$ is an open neighbourhood of x which doesn't intersect with $\bigcup_{A\in\mathcal{A}} A$ and thus $x \notin \overline{\bigcup_{A\in\mathcal{A}} A}$. The result follows. □

Back to the proof.

---

[3] Note: here we don't require elements of $\mathscr{A}$ to be open

(1) [Paracompact + (T2) $\Longrightarrow$ (T3)]
Suppose $x \in X$ and V is an open neighborhood of x. Then for any $y \in V^c$, there exist open sets $U_y \ni x, V_y \ni y$ such that $U_y \cap V_y = \emptyset$. Now

$$\mathscr{U}_1 := \{V_y \mid y \in V^c\} \cup \{V\}$$

is an open covering of X, so it has a locally finite refinement $\widetilde{\mathscr{U}}$. Let

$$\mathscr{U} = \{W \in \widetilde{\mathscr{U}} \mid \exists y \in V^c \text{ s.t. } W \subset V_y\}.$$

Then $\mathscr{U}$ is a locally finite covering of V^c. Take

$$U = \overline{\bigcup_{W \in \mathscr{U}} W}^{\,c}.$$

Then we have $x \in U$ since

$$x \notin \bigcup_{W \in \mathscr{U}} \overline{W} = \overline{\bigcup_{W \in \mathscr{U}} W},$$

and we also have $\overline{U} \subset V$ since

$$V^c \subset \bigcup_{W \in \mathscr{U}} W \subset \overset{\circ}{\overline{\bigcup_{W \in \mathscr{U}} W}} = \overline{\overline{\bigcup_{W \in \mathscr{U}} W}^{\,c}}^{\,c} = \overline{U}^c.$$

It follows from the equivalent characterization that X is (T3).

(2) [Paracompact + (T3) $\Longrightarrow$ (T4)]
Repeat the proof above word by word, with "a point x" replaced by "a closed subset A" and "(Ti)" replaced by "(Ti+1)".

$\square$

Now we can state

Theorem 2.3 (Existence of P.O.U.)**.** *Let X be paracompact and Hausdorff. Then for any open covering $\{U_\alpha\}$ of X, there is a partition of unity subordinate to $\{U_\alpha\}$.*

Proof.

> **Idea:** We want to apply the preliminary version of P.O.U. construction that we proved last time. So we need a locally finite open covering $\{U_\alpha\}$, and closed sets $K_\alpha \subset U_\alpha$ such that $\{K_\alpha\}$ is also a covering of X. In other words, we need to "shrink" the open covering $\{U_\alpha\}$ to a smaller "closed covering".
>
> If you are careful enough, you might have seen that "shrinking U_α to K_α and apply the preliminary version" is still not quite enough, because what we want is $\text{supp}(\rho_\alpha) \subset U_\alpha$, which is stronger than $\rho_\alpha = 0$ on U_α^c in the conclusion of the preliminary version. [And this is the reason I call that one a "preliminary" version.]

Now we prove the theorem.

Step 1. We first prove a "shrinking lemma".

Lemma 2.4. *Let X be paracompact and Hausdorff, and $\mathscr{U} = \{U_\alpha\}$ be an open covering of X. Then there exists a locally finite open refinement $\mathscr{V} = \{V_\alpha\}$* [4] *of $\mathscr{U}$ such that $\overline{V_\alpha} \subset U_\alpha$ for any α.*

Proof. Let

$$\mathscr{A} = \{A \in \mathscr{T} \mid \exists U_\alpha \text{ s.t. } \overline{A} \subset U_\alpha\}.$$

Since X is paracompact and (T2), it is also (T3) and (T4). So $\mathscr{A}$ is an open covering of X. Let

$$\mathscr{B} = \{B_\beta \mid \beta \in \Lambda\}$$

be a locally finite open refinement of $\mathscr{A}$, where the index set could be different from that of $\mathscr{A}$. For each β, we choose $\alpha = f(\beta)$ such that

$$\overline{B_\beta} \subset U_{f(\beta)}.$$

Now for each index α of the family $\mathscr{U}$, let

$$V_\alpha = \bigcup_{f(\beta)=\alpha} B_\beta,$$

where we take $V_\alpha = \emptyset$ if no such β exists. According to Lemma 2.2,

$$\overline{V_\alpha} = \overline{\bigcup_{f(\beta)=\alpha} B_\beta} = \bigcup_{f(\beta)=\alpha} \overline{B_\beta} \subset U_\alpha.$$

It remains to check local finiteness: For any $x \in X$, there exists an open neighborhood U_x of x which intersects with only finitely many B_β's. As a result, U_x only intersects with those α such that $f(\beta) = \alpha$. $\square$

Step 2. Construct P.O.U.

Apply the "shrinking lemma" that we just proved three times to get two locally finite open coverings $\{V_\alpha\}$, $\{W_\alpha\}$, and $\{Z_\alpha\}$ such that

$$\overline{W_\alpha} \subset V_\alpha \subset \overline{V_\alpha} \subset Z_\alpha \subset \overline{Z_\alpha} \subset U_\alpha.$$

Now apply the preliminary P.O.U. theorem that we proved last time for $\overline{W_\alpha} \subset V_\alpha$, to get continuous functions $\rho_\alpha : X \to [0,1]$ such that:

- $\rho_\alpha > 0$ on $\overline{W_\alpha}$,
- $\rho_\alpha = 0$ on V_α^c,
- $\sum_\alpha \rho_\alpha = 1$ on X.

---

[4] Note: this is a very strong refinement since $\{V_\alpha\}$ and $\{U_\alpha\}$ has the same set of indices.

The family $\{\rho_\alpha\}$ is a P.O.U. subordinate to $\{U_\alpha\}$, since

$$\operatorname{supp} \rho_\alpha \subset \overline{V_\alpha} \subset U_\alpha,$$

and $\{\operatorname{supp} \rho_\alpha\}$ is locally finite since $\operatorname{supp}(\rho_\alpha) \subset \overline{V_\alpha} \subset Z_\alpha$, and the latter is locally finite. □

Remark. In fact, for a Hausdorff space, any open covering admits such an P.O.U. if and only if it is paracompact.

Remark. As we mentioned above, manifolds are topological spaces which are (A2), (T2) and locally Euclidean, and thus are always paracompact. Geometric objects like sphere, torus, $\mathbb{RP}^n$ etc are all manifolds. Since they are locally Euclidean, locally we can pretend that we are working on Euclidian spaces (via coordinates). With partition of unity at hand, you can glue these local data to a global one, e.g.

- glue locally defined continuous functions to global continuous functions,
- first define integrals locally and then glue to define global integrals,
- glue locally defined vector fields to global vector fields,
- glue locally defined "inner product structure" to global Riemannian metric

3. The Nagata-Smirnov Metrization Theorem

Paracompactness (or more precisely, local finiteness) is used not only for constructing P.O.U., but also for characterizing metrizability. Characterizing which topological space is metrizable was a big problem in general topology. The first big step in searching for a solution to this problem is Urysohn metrization theorem which claims that any topological space satisfying (A2), (T2) and (T4) is metrizable. Of course since (T2) implies (T1), we can replace the condition (T4) by (T3). So these separation axioms are necessary conditions for a space to be metrizable. However, the countability assumption (A2) is not necessary. So to find a characterization of metrizability, a natural idea is to find a weaker condition replacing (A2). The correct condition was finally found by Nagata and Smirnov around 1950 independently, which is the one we introduced a while ago: σ-locally finite.

Theorem 3.1 (Nagata-Smirnov). *A topological space $(X, \mathscr{T})$ is metrizable if and only if it is (T2), (T3) and admits a base which is σ-locally finite.*

Proof of Nagata-Smirnov metrization theorem.

First suppose X is metrizable. Then it is (T2) and (T4), and thus is (T1) and (T3). It is paracompact by Stone's theorem. So for each n, the open covering

$$\mathscr{U}_n = \{B(x, \frac{1}{n}) \mid x \in X\}$$

has a locally finite open refinement $\mathscr{B}_n$. It remains to prove that the σ-locally finite collection $\mathscr{B} = \cup_n \mathscr{B}_n$ is a base, which is standard: For any $x \in X$ and any $\varepsilon > 0$, we choose n such that $\frac{1}{n} < \frac{\varepsilon}{2}$. There is an open set B in $\mathscr{B}_n$ such that $x \in B$. It follows $B \subset B(x, \frac{2}{n}) \subset B(x, \varepsilon)$.

Conversely suppose X is (T2), (T3) and admits a σ-locally finite base.

> **Lemma 3.2.** *Suppose X is (T3) and has a σ-locally finite base. Then*
> *(1) Any closed set in X is a G_δ-set.*
> *(2) X is (T4).*
>
> So in some sense the existence of "σ-locally finite base" is a version of countability, which enhances separability [similar as (A2) or Lindelf̈]. We will leave the proof of the lemma as an exercise.

Let's continue our proof.

According to the lemma, X is (T4) and any closed subset in X is a G_δ-set. Let $\mathscr{B} = \cup_n \mathscr{B}_n$ be a σ-locally finite base, where each $\mathscr{B}_n$ is locally finite. For any $B \in \mathscr{B}_n$, we choose a continuous function $f_{n,B} : X \to [0, 1/n]$ such that

$$f_{n,B}^{-1}(0) = B^c.$$

Now define

$$d_n(x,y) = \sum_{B \in \mathscr{B}_n} |f_{n,B}(x) - f_{n,B}(y)|.$$

It is a continuous function since the summation is locally finite. By definition the function d_n satisfies $d_n(x,y) = d_n(y,x)$ and $d_n(x,z) \le d_n(x,y) + d_n(y,z)$. However, it is not a metric, because it is not point-separating in general: we may have $d_n(x,y) = 0$ for some $x \neq y$.[5] The good news is that we have plenty of d_n's which are enough to separate points: In fact, we have

Fact. For any closed set F and $x \notin F$, there exists n and $B \in \mathscr{B}_n$ such that

$$d_n(x,y) \ge a := f_{n,B}(x) > 0, \quad \forall y \in F.$$

> To see this we just take n and $B \in \mathscr{B}_n$ such that $x \in B \subset F^c$. Then $f_{n,B}(x) > 0$ and $f_{n,B}(F) = 0$. So $d_n(x,y) \ge f_{n,B}(x) > 0$ for all $y \in F$.

In particular, for $y \neq x$ if we take $F = \{y\}$, we get $d_n(x,y) > 0$.

Now we define

$$d(x,y) = \sum_n 2^{-n} d_n(x,y).$$

Then d is a metric on X. It remains to prove that the metric topology coincides with $\mathscr{T}$. Since d is continuous with respect to the topology $\mathscr{T}$, any metric ball is open in $\mathscr{T}$. It follows $\mathscr{T}_d \subset \mathscr{T}$. Conversely, for any open set $U \in \mathscr{T}$ and any $x \in U$, according to the fact we just proved, we can find n and $B \in \mathscr{B}_n$ such that $d(x,y) > r = 2^{-n} f_{n,B}(x) > 0$ for all $y \in U^c$, i.e. $B(x,r) \subset U$. It follows that $U \in \mathscr{T}_d$. So $\mathscr{T} = \mathscr{T}_d$. $\square$

---

[5] Such a structure is called a *pseudo-metric*.