Topology (H) Lecture 14
Lecturer: Zuoqin Wang
Time: May 9, 2020

THE ARZELA-ASCOLI THEOREM

Last time we learned:

- Three topologies on $\mathcal{C}(X, \mathbb{R})$
- Stone-Weierstrass theorem: various versions and generalizations

1. Four topologies on $\mathcal{C}(X, Y)$

Let X be a set and (Y, d) be a metric space. As we did last time, we can easily define three topologies on the space $\mathcal{M}(X, Y) = Y^X$ of all maps form X to Y:

$$\mathscr{T}_{product} = \mathscr{T}_{p.c.} \subset \mathscr{T}_{u.c.} \subset \mathscr{T}_{box},$$

where the uniform topology $\mathscr{T}_{u.c.}$ is generated by the uniform metric

$$d_u(f, g) = \sup_{x \in X} \frac{d(f(x), g(x))}{1 + d(f(x), g(x))}$$

which characterizes the "uniform convergence" of sequence of maps. It is not hard to extend Proposition 1.2 in lecture 13 to this slightly more general setting[PSet 5-2-1(a)]:

Proposition 1.1. *If d is complete on Y, then the uniform metric d_u is a complete metric on $\mathcal{M}(X, Y)$.*

Again for the subspaces containing only "bounded maps" (i.e. maps whose images are in a fixed bounded subset in Y), we can replace d_u with a slightly simpler one:

$$d_u(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

Now we assume X is a topological space, so that we can talk about the continuity of maps from X to Y. Then the three topologies alluded to above induce three topologies on the subspace

$$\mathcal{C}(X, Y) = \{f \in \mathcal{M}(X, Y) \mid f \text{ is continuous}\}.$$

We want to find a reasonable topology on $\mathcal{C}(X, Y)$ so that "bad convergent sequences" are no longer convergent in this topology, while "good convergent sequences" are still convergent. Unfortunately, none of the three topologies above behaves perfect with this purpose. For example, we may take $X = Y = \mathbb{R}$, then

- If we use the pointwise convergence topology, then the sequence of functions $f_n(x) = e^{-nx^2}$ converges in $\mathscr{T}_{p.c.}$ to a bad limit function, the discontinuous function $f_0(x)$ which equals 1 at $x = 0$ and equals 0 for all other x. This is because

  The pointwise convergence topology (=the product topology) is too weak for the limit of a convergent sequence of continuous functions to be continuous.

- If we use the uniform convergence topology, then the sequence of functions $f_n(x) = \frac{x^2}{n}$ would not converge in $\mathscr{T}_{u.c.}$, although it does converge to a nice limit function $f_0(x) \equiv 0$ in the pointwise sense. This is because

  The uniform topology (and thus the box topology) is too strong for a sequence to converge.

So it should be good if there is a new topology on $\mathcal{C}(X,Y)$ that is weaker than $\mathscr{T}_{u.c.}$, but the limit of a convergent sequence of continuous functions with respect to this new topology is still continuous. The answer is yes, which is based on a very simple observation:

> Continuity is a "local phenomena" (which is stronger than "pointwise phenomena" and weaker than "global phenomena"). The pointwise convergence is too weak since it is a pointwise conception. The uniform convergence is too strong since it is a global conception. So the correct way is to replace the uniform convergence by its local analogue.

For example, although $f_n(x) = \frac{x^2}{n} \nrightarrow f(x) = 0$ uniformly on $\mathbb{R}$, we do have:

$$\forall [a,b] \subset \mathbb{R}, f_n(x) = \frac{x^2}{n} \longrightarrow f(x) = 0 \ \underline{\text{uniformly}} \text{ on } [a,b].$$

In general, let X be a topological space, (Y,d) be a metric space. For any compact set $K \subset X$ and any $\varepsilon > 0$, we denote [Compare: $\omega(f; x_1, \cdots, x_n; \varepsilon)$ in Lecture 3!]

$$B(f; K, \varepsilon) = \{g \in \mathcal{M}(X,Y) \mid \sup_{x \in K} d(f(x), g(x)) < \varepsilon\}.$$

Lemma 1.2. *The family*

$$\mathscr{B}_{c.c.} = \{B(f; K, \varepsilon) \mid f \in \mathcal{M}(X,Y), K \subset X \text{ compact}, \varepsilon > 0\}$$

is a base of a topology $\mathscr{T}_{c.c}$ on $\mathcal{M}(X,Y)$, which satisfies the following property:

$$\boxed{f_n \to f \text{ uniformly on each compact set in } X \iff f_n \to f \text{ in } (\mathcal{M}(X,Y), \mathscr{T}_{c.c.}).}$$

Proof. The family $\mathscr{B}_{c.c.}$ is a base because for any

$$g \in B(f_1; K_1, \varepsilon_1) \cap B(f_2; K_2, \varepsilon_2),$$

if we take

$$\varepsilon_0 = \min(\varepsilon_1 - \sup_{x \in K_1} d(f_1(x), g(x)),\ \varepsilon_2 - \sup_{x \in K_2} d(f_2(x), g(x))),$$

then we have

$$B(g; K_1 \cap K_2, \varepsilon_0) \subset B(f_1; K_1, \varepsilon_1) \cap B(f_2; K_2, \varepsilon_2),$$

The topology $\mathscr{T}_{c.c.}$ satisfies the demanded property because

$$\begin{aligned} & f_n \to f \text{ uniformly on each compact subset } K \subset X \\ \Longleftrightarrow & \forall \varepsilon > 0, \forall \text{ compact } K \subset X, \exists N \text{ s.t. } \sup_{x \in K} d(f_n(x), f(x)) < \varepsilon, \forall n > N \\ \Longleftrightarrow & \forall \varepsilon > 0, \forall \text{ compact } K \subset X, \exists N \text{ s.t. } f_n \in B(f; K, \varepsilon), \forall n > N \\ \Longleftrightarrow & f_n \to f \text{ in } (\mathcal{M}(X, Y), \mathscr{T}_{c.c.}). \end{aligned}$$

□

Definition 1.3. The topology $\mathscr{T}_{c.c.}$ on $\mathcal{M}(X, Y)$ generated by the base $\mathscr{B}_{c.c.}$ is called the *compact convergence topology.*

Remark.

(1) By definition, we always have $\mathscr{T}_{product} = \mathscr{T}_{p.c.} \subset \mathscr{T}_{c.c.} \subset \mathscr{T}_{u.c.}$ on $\mathcal{M}(X, Y)$.
(2) If X is compact, then $\mathscr{T}_{c.c.} = \mathscr{T}_{u.c.}$.
(3) Let $A \subset X$ be any subset. It is easy to check that the restriction map

$$r : \mathcal{M}(X, Y) \to \mathcal{M}(A, Y), \quad f \mapsto f|_A$$

is continuous with respect to all three topologies. Since the restriction of a continuous map to a subset is still continuous, the restriction map

$$r : \mathcal{C}(X, Y) \to \mathcal{C}(A, Y), \quad f \mapsto f|_A$$

is continuous with respect to all three topologies: $\mathscr{T}_{p.c.}, \mathscr{T}_{c.c.}, \mathscr{T}_{u.c.}$.

Under some very weak conditions, the limit of a sequence of continuous functions (with respect to $\mathscr{T}_{c.c}$) is continuous:

Proposition 1.4. *If X is locally compact, Y is a metric space, then $\mathcal{C}(X, Y)$ is closed in $(\mathcal{M}(X, Y), \mathscr{T}_{c.c.})$. In particular, if $f_n \in \mathcal{C}(X, Y)$ converges to f in $(\mathcal{M}(X, Y), \mathscr{T}_{c.c.})$, then $f \in \mathcal{C}(X, Y)$.*

Proof. Let $f \in \mathcal{M}(X, Y)$ be a limit point of $\mathcal{C}(X, Y)$ with respect to $\mathscr{T}_{c.c.}$. We need to prove f is continuous at any $x \in X$. By the definition of local compactness, for any $x \in X$, there exists an open neighbourhood U of x and a compact set K such that $U \subset K$. Since f is a limit point, there exists $f_n \in \mathcal{C}(X, Y) \cap B(f, K, \frac{1}{n})$. It follows that f_n converges uniformly on K to f. So f is continuous on K. In particular, it is continuous on U and thus continuous at x. □

Remark. We can replace the "local compactness" assumption by the "first countability" assumption (A1), so in particular the same proposition holds if X is a metric space.

The compact convergence topology is defined for maps from a topological space to a metric space. It is easy to extend the definition of compact convergence topology to a topology on $\mathcal{M}(X,Y)$, where both X and Y are topological spaces:

Definition 1.5. Let X, Y be topological spaces. For any compact $K \subset X$ and open $V \subset Y$, we denote

$$S(K,V) = \{f \in \mathcal{M}(X,Y) \mid f(K) \subset V\}.$$

The topology $\mathscr{T}_{c.o.}$ on $\mathcal{M}(X,Y)$ generated by the sub-base

$$\mathscr{S}_{c.o.} = \{S(K,V) \mid K \subset X \text{ compact }, V \subset Y \text{ open }\}$$

is called the *compact-open topology.*[1]

We are only interested in $\mathscr{T}_{c.o.}$ on $\mathcal{C}(X,Y)$.

Remark.

(1) For example, if we take X be a single point set $\{*\}$, then the space $(\mathcal{C}(\{*\},Y)$ is homeomorphic to the space Y itself.
(2) One can prove [Today's PSet] that if Y is a metric space, then $\mathscr{T}_{c.o.} = \mathscr{T}_{c.c.}$ on $\mathcal{C}(X,Y)$. In particular,
    - $\mathscr{T}_{c.c.}$ on $\mathcal{C}(X,Y)$ is independent of the metric on Y.
    - If X is compact, then $\mathscr{T}_{u.c.}$ on $\mathcal{C}(X,Y)$ is independent of the metric on Y.

It turns out that with respect to the compact-open topology, the composition is continuous as long as the "middle variable space" is locally compact:

Proposition 1.6. *Suppose X, Y and Z are topological spaces, where Y is locally compact Hausdorff. Then the composition map*

$$\circ : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z), \quad (f,g) \mapsto g \circ f$$

is continuous (with respect to the compact-open topologies on each space).

Proof. It is enough to check the pre-image of any element is a sub-base is open. So we take any $S(K_X, V_Z)$, where K_X is compact in X and V_Z is open in Z. Choose any $g \circ f \in S(K_X, V_Z)$. By continuity of f and g, $f(K_X)$ is compact in Y, $g^{-1}(V_Z)$ is open in Y, and $f(K_X) \subset g^{-1}(V_Z)$. Since Y is locally compact, by Lemma 2.4 in Lecture 11, there exists an open set U_Y in Y with $\overline{U_Y}$ compact such that

$$f(K_X) \subset U_Y \subset \overline{U_Y} \subset g^{-1}(V_Z).$$

It follows that for any $f_1 \in S(K_X, U_Y)$ and any $g_1 \in S(\overline{U_Y}, V_Z)$, one has $g_1 \circ f_1 \in S(K_X, V_Z)$. This completes the proof. □

[1]The compact-open topology can be defined on the whole space $\mathcal{M}(X,Y)$ for topological spaces X and Y, but it is most useful on the subspace $\mathcal{C}(X,Y)$.

Corollary 1.7. *Let X be a locally compact Hausdorff space, and Y be any topological space. Then the evaluation map*

$$e : X \times \mathcal{C}(X,Y) \to Y, \quad (x,f) \mapsto e(x,f) = f(x) \in Y$$

is continuous when we endow $\mathcal{C}(X,Y)$ with the compact-open topology.

Proof. Identifying X with $\mathcal{C}(\{*\},X)$ and Y with $\mathcal{C}(\{*\},Y)$. Then the evaluation map is just the composition map. □

2. Arzela-Ascoli theorem

Given a sequence, or more generally, a family of continuous functions. One of the central problem in analysis is: Can one find a subsequence that converges (uniformly) to another continuous function?

For example, in analysis, to prove the existence of a solution to a PDE or a variational problem, one can first try construct a sequence of functions which solve the problem "approximately". If one can show the sequence of "approximate solutions" has a subsequence that converges to a nice function, then usually with some extra work, one can show that the limit would be a true solution. Such a method is known as a "compactness argument". One of the most useful tool to carry out such a compactness argument for functions is the Arzela-Ascoli theorem. As we mentioned in Lecture 1, one motivation for the Arzela-Ascoli theorem is to rescue Dirichlet's principle, i.e. trying to prove the existence of a solution to the Laplace equation $\Delta u(x,y,z) = 0$ with prescribed boundary conditions.

The original version of Arzela-Ascoli theorem that you may have seen in your analysis course is

Theorem 2.1 (Arzela-Ascoli, classical version)**.** *A sequence $\{f_n\} \in \mathcal{C}([0,1],\mathbb{R})$ has a convergence subsequence if and only if it is uniformly bounded and equicontinuous.*

Recall that a family of functions $\mathcal{F} \subset \mathcal{C}([0,1],\mathbb{R})$ is

- *uniformly bounded* if there exists $M > 0$ such that for all $x \in [0,1]$ and all $f \in \mathcal{F}$, we have
$$|f_n(x)| \le M.$$
- *equicontinuity* if for any $x \in [0,1]$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in [0,1]$ with $|y - x| < \delta$ and for all $f \in \mathcal{F}$, we have
$$|f(y) - f(x)| \le \varepsilon.$$

It is not hard to see that the two conditions are necessary:

(1) The sequence $f_n(x) = n$ is equicontinuous but has no convergent subsequence because it is not uniformly bounded (although each function in the sequence is a bounded function).

(2) The sequence $f_n(x) = x^n$ is uniformly bounded on $[0, 1]$ but has no convergent subsequence (in $\mathcal{C}([0, 1], \mathbb{R})$ because it is not equicontinuous at $x = 1$ (although each function in the sequence is continuous at $x = 1$).

The conception of equicontinuity can be easily generalized to maps from an arbitrary topological space X to a metric space Y:

Definition 2.2. Let (Y, d) be a metric space, and X be a topological space. Let $\mathcal{F} \subset \mathcal{C}(X, Y)$ be a subset.

(1) We say $\mathcal{F}$ is *equicontinuous* at $x_0 \in X$ if for any $\varepsilon > 0$, there exists an open neighbourhood U of x_0 such that
$$d(f(x), f(x_0)) < \varepsilon, \quad \forall x \in U, \forall f \in \mathcal{F}.$$
(2) We say $\mathcal{F}$ is *equicontinuous* if it is equicontinuous at any point $x \in X$.

Note that equicontinuity is a metric property: it depends on the metric on Y. It turns out that equicontinuity is a generalization of totally boundedness in $(\mathcal{C}(X, Y), d_u)$:

Proposition 2.3. *Let (Y, d) be a metric space, and $\mathcal{F}$ be a totally bounded subset in $\mathcal{C}(X, Y)$ (with respect to d_u). Then $\mathcal{F}$ is equicontinuous.*

Proof. For any $x_0 \in X$ and $\varepsilon > 0$, we need to find an open neighborhood U of x_0 s.t.
$$d(f(x), f(x_0)) < \varepsilon, \quad \forall x \in U, \forall f \in \mathcal{F}.$$
Since $\mathcal{F}$ is totally bounded, there exists a finite $\frac{\varepsilon}{4}$-net $\{f_1, \cdots, f_n\}$ of $\mathcal{F}$ in $(\mathcal{C}(X, Y), d_u)$. Since each f_k is continuous, the set
$$U = \bigcap_{k=1}^{n} f_k^{-1}\left(B(f_k(x_0), \frac{\varepsilon}{3})\right)$$
is an open neighborhood of x_0. Now for any $f \in \mathcal{F}$, by our choice there exists k such that $d_u(f, f_k) < \varepsilon/4$. It follows that for any $x \in U$,
$$d(f(x), f(x_0)) \leq d(f(x), f_k(x)) + d(f_k(x), f_k(x_0)) + d(f_k(x_0), f(x_0)) < \varepsilon.$$
This completes the proof. □

We want $\mathcal{F}$ to satisfy the property that any sequence in $\mathcal{F}$ has a (uniformly) convergent subsequence which converges to a function in $\mathcal{C}(X, Y)$ (but the limit may be outside $\mathcal{F}$). In other words, we want the closure $\overline{\mathcal{F}}$ to be compact or is contained in a compact set in $\mathcal{C}(X, Y)$ with respect to $\mathscr{T}_{c.c.}$ (or $\mathscr{T}_{u.c.}$ if you want uniform convergence).

Definition 2.4. A subset A in a topological space X is called *precompact* (or *relatively compact*) if $\overline{A}$ is compact.

For simplicity we also introduce the following definition:

Definition 2.5. We say a family $\mathcal{F} \subset \mathcal{C}(X, Y)$ is

(1) *pointwise bounded* if $\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\}$ is bounded in Y for each $a \in X$,
(2) *pointwise precompact* if $\mathcal{F}_a$ is precompact in Y for each $a \in X$.

Today we are going to prove the following very general form[2] of Arzela-Ascoli theorem:

Theorem 2.6 (Arzela-Ascoli theorem, the general version)**.** *Let X be a topological space and (Y, d) a metric space. Let $\mathcal{F}$ be a subset of $\mathcal{C}(X, Y)$ which is endowed with the compact-convergence topology $\mathscr{T}_{c.c.}$.*

(1) Suppose $\mathcal{F}$ is equicontinuous and pointwise precompact. Then the closure of $\mathcal{F}$ is compact in $(\mathcal{C}(X, Y), \mathscr{T}_{c.c.})$.
(2) If X is locally compact and Hausdorff, then the converse holds.

Note that the conclusion is quite weak in this very general version, because in general the topology $\mathscr{T}_{c.c.}$ need to be metrizable, and compactness does not imply sequential compactness. Thus for a sequence which is equicontinuous and pointwise precompact, we can't even conclude the existence of a convergent subsequence. However, there are many interesting special cases where we are able to conclude the existence of a convergent subsequence:

(a) We know that if X is compact and Y is a metric space, then $\mathscr{T}_{c.c.} = \mathscr{T}_{u.c}$ on $\mathcal{C}(X, Y)$. Since $\mathscr{T}_{u.c.}$ is a metric topology, compactness does imply sequential compactness. So in particular we get

Theorem 2.7 (Arzela-Ascoli for maps on compact spaces)**.** *Let X be compact and (Y, d) a metric space. Let $\mathcal{F} \subset \mathcal{C}(X, Y)$ be a subset which is equicontinuous and pointwise precompact. Then any sequence in $\mathcal{F}$ has a subsequence that converges* uniformly *(to a continuous map) on X.*

Since in $\mathbb{R}^n$, a set is pre-compact if and only if it is bounded, we get

Corollary 2.8 (Arzela-Ascoli for functions on compact spaces)**.** *Let X be compact. Let $\mathcal{F} \subset \mathcal{C}(X, \mathbb{R}^n)$ be a subset which is equicontinuous and pointwise bounded. Then any sequence in $\mathcal{F}$ has a subsequence that converges uniformly (to a bounded continuous function) on X.*

(b) For the case of locally compact space, every point has a compact neighborhood. Obviously if a family $\mathcal{F}$ is equicontinuous/pointwise precompact, then its restriction to such a neighborhood is also equicontinuous/pointwise precompact. So if X is locally compact, then for any sequence $\{f_n\}$ which is equicontinuous and pointwise compact, and for any point x, there is a compact neighborhood of x one which $\{f_n\}$ has a convergent subsequence. Unfortunately this is not strong enough to claim that the sequence $\{f_n\}$ has a convergent subsequence in $\mathscr{T}_{c.c}$, because there may

[2]There exists even more general form of Arzela-Ascoli theorem which characterize the compactness of family of maps into a uniform space (which is a generalization of metric space).

have "too much compact subsets" in X. However, if we assume X is σ-compact, i.e. X is a countably union of compact subsets, then we may apply the standard diagonalization trick to extract a subsequence which converges (uniformly!) on each compact subset:

Theorem 2.9 (Arzela-Ascoli for maps on locally compact and σ-compact spaces)**.** *Let X be locally compact and σ-compact, and (Y,d) be a metric space. Let $\mathcal{F} \subset \mathcal{C}(X,Y)$ be a subset which is equicontinuous and pointwise precompact. Then any sequence in $\mathcal{F}$ has a subsequence that converges uniformly on compact sets of X to a limit function $f \in \mathcal{C}(X,Y)$.*

Now let's prove the main theorem. Although the theorem is about the compact convergence topology $\mathscr{T}_{c.c.}$, we will use $\mathscr{T}_{p.c.}$ and $\mathscr{T}_{u.c}$ in the proof as well.

Proof of Theorem 2.6.

(1) **Idea:** We want to prove that $\overline{\mathcal{F}}$ is a compact set in $(\mathcal{C}(X,Y), \mathscr{T}_{c.c.})$. The idea is to find another space in which the closure of $\mathcal{F}$ is compact. Then try to prove the two topologies on the closure of $\mathcal{F}$ are the same. (The auxiliary space should have less open sets such that it is easier for a set to be compact. The best candidate is the product topology, since we have the wonderful Tychonoff theorem.)

Notation: Although as a set we have $\mathcal{M}(X,Y) = Y^X$, we will distinguish these two notions in this proof: we write $\mathcal{M}(X,Y)$ or $\mathcal{C}(X,Y)$ we will use the compact convergence topology, and when we write Y^X we will use the product topology, i.e. the pointwise convergence topology. So in this proof, $\mathcal{C}(X,Y)$ is NOT a topological subspace of Y^X, although it is a subset.

We denote $\mathcal{K}$ = the closure of $\mathcal{F}$ in Y^X.

Step1: $\mathcal{K}$ is compact in Y^X.

Let $K_a = \overline{\mathcal{F}_a}$ in Y. Then K_a is compact and closed. So

$$\prod_{a\in X} K_a = \bigcap_{a\in X} \pi_a^{-1}(K_a)$$

is compact (by Tychonoff) and closed in Y^X. Since

$$\mathcal{F} \subset \prod_{a\in X} \mathcal{F}_a \subset \prod_{a\in X} K_a,$$

its closure $\mathcal{K}$, as a closed subset in the compact set $\prod_{a\in X} K_a$, is compact in Y^X.

Step2: $\mathcal{K}$ is equicontinuous (in particular, $\mathcal{K}$ is subset of $\mathcal{C}(X,Y)$).

For any $x_0 \in X$ and $\varepsilon > 0$, we need an open neighborhood U of x_0 s.t.

(*) $$d(g(x), g(x_0)) < \varepsilon, \quad \forall x \in U,\ \forall g \in \mathcal{K}.$$

By the equicontinuity of $\mathcal{F}$, we can find an open neighbourhood U of x_0 s.t.

$$d(f(x), f(x_0)) < \frac{\varepsilon}{3}, \quad \forall x \in U,\ \forall f \in \mathcal{F}.$$

To prove (*) for this U, we fix any $g \in \mathcal{K}$, $x \in U$ and denote

$$\begin{aligned} V &= \left\{ h \in Y^X \,\middle|\, d(h(x), g(x)) < \frac{\varepsilon}{3},\ d(h(x_0), g(x_0)) < \frac{\varepsilon}{3} \right\} \\ &= \pi_x^{-1}\left((g(x) - \frac{\varepsilon}{3}, g(x) + \frac{\varepsilon}{3}) \right) \bigcap \pi_{x_0}^{-1}\left((g(x_0) - \frac{\varepsilon}{3}, g(x_0) + \frac{\varepsilon}{3}) \right). \end{aligned}$$

Then V is an open neighbourhood of g in Y^X. Since $g \in \mathcal{K}$ and $\mathcal{K}$ is the closure of $\mathcal{F}$ in Y^X, we have $V \cap \mathcal{F} \neq \emptyset$. Take any $f \in V \cap \mathcal{F}$,

$$d(g(x), g(x_0)) \leqslant d(g(x), f(x)) + d(f(x), f(x_0)) + d(f(x_0), g(x_0)) < \varepsilon.$$

This proves (*) and thus the equicontinuity of $\mathcal{K}$.

Step3: The two topologies $\mathscr{T}_{product}$ and $\mathscr{T}_{c.c.}$ coincide on $\mathcal{K}$.

[Once this is done, then together with Step 1, we conclude that $\mathcal{K}$ is the closure of $\mathcal{F}$ in $(\mathcal{C}(X,Y), \mathscr{T}_{c.c.})$ and is compact. This proves (1).]

Since we always have $\mathscr{T}_{product} \subset \mathscr{T}_{c.c.}$, it is enough to prove the reverse on $\mathcal{K}$. In other words, we need to show: for any $g \in \mathcal{K}$ and for any $B(g; K, \varepsilon) \subset \mathcal{C}(X,Y)$, where $K \subset X$ is compact, there exists an open set $U \subset Y^X$ such that

(**) $$U \cap \mathcal{K} \subset B(g; K, \varepsilon) \cap \mathcal{K}.$$

Since K is compact in X, and $\mathcal{K}$ is equicontinuous, we can find finitely many points $x_1, \cdots, x_n$ and open sets $V_1, \cdots, V_n$ in X covering K, such that

$$d(\tilde{g}(x), \tilde{g}(x_i)) < \frac{\varepsilon}{3}, \quad \forall \tilde{g} \in \mathcal{K},\ \forall x \in V_i.$$

So we take U to be the set

$$U = \omega(g; x_1, \cdots, x_n, \varepsilon) = \left\{ h \in Y^X | d(h(x_i), g(x_i)) < \frac{\varepsilon}{3}, 1 \leqslant i \leqslant n \right\}.$$

It is easy to check (**) holds for this U:

If $h \in U \cap \mathcal{K}$, then for any $x \in K$, there exists i s.t. $x \in V_i$. So

$$\begin{aligned} d(h(x), g(x)) &\leqslant d(h(x), h(x_i)) + d(h(x_i), g(x_i)) + d(g(x_i), g(x)) \\ &< \varepsilon, \quad \forall x \in K. \end{aligned}$$

In other words, $h \in B(g; K, \varepsilon)$.

This completes Step 3 and thus proves (1).

(2) Now suppose X is LCH, and suppose the closure $\mathcal{K}$ of $\mathcal{F}$ in $\mathcal{C}(X,Y)$ is compact. We will show $\mathcal{K}$ is equicontinuous and pointwise compact, which would imply that $\mathcal{F}$ is equicontinuous and pointwise pre-compact (since each $\overline{\mathcal{F}_a}$ is a closed subset in $\mathcal{K}_a$).

The compactness of $\mathcal{K}_a$ follows from Corollary 1.7: $\mathcal{K}_a$ is the image of the compact set $\mathcal{K}$ under the continuous map

$$\mathcal{C}(X,Y) \xrightarrow{j_a} X \times \mathcal{C}(X,Y) \xrightarrow{e} Y$$

and thus is compact, where j_a is the "embedding map" $j_a(f) := (a, f)$.

To show the equicontinuity of $\mathcal{K}$ at an arbitrary $x \in X$, we take a compact neighborhood A of x. Then it is enough to show

$$\mathcal{K}_A := \{r(f) \mid f \in \mathcal{K}\}$$

is equicontinuous at x, where $r : \mathcal{C}(X,Y) \to \mathcal{C}(A,Y)$ is the restriction map. Since r is continuous, $\mathcal{K}_A = r(\mathcal{K})$ is compact in $\mathcal{C}(A,Y)$. Since A is compact, the compact convergence topology coincides with the uniform convergence topology on $\mathcal{C}(A,Y)$. So compactness of $\mathcal{K}_A$ in $\mathcal{C}(A,Y)$ implies that $\mathcal{K}_A$ is totally bounded with respect to d_u. By Proposition 2.3, $\mathcal{K}_A$ is equicontinuous. This completes the proof.

□

Remark. In proving (2) we only used a weaker condition: $\mathcal{F}$ is contained in a compact set $\mathcal{K}$ in $\mathcal{C}(X,Y)$.

3. Applications of Arzela-Ascoli theorem

The Arzela-Ascoli theorem is widely used in analysis. Here are some standard applications that you can learn from other courses:

- Functional analysis: Frechet-Kolmogorov-Riesz compactness theorem.
- PDE: Sobolev embedding etc.
- ODE: Peano existence theorem.
- Complex analysis: Montel's theorem
- Harmonic analysis: Peter-Weyl theorem.

We end this lecture by giving an application to convex geometry. Recall that a subset $A \subset \mathbb{R}^n$ is *convex* if

$$x, y \in A \Longrightarrow (1-\lambda)x + \lambda y \in A, \quad \forall\ 0 \leqslant \lambda \leqslant 1.$$

In what follows we consider

$$\mathfrak{C}(\mathbb{R}^n) = \text{ the set of all non-empty compact convex subsets in } \mathbb{R}^n.$$

Note that $\mathfrak{C}(\mathbb{R}^n)$ is a subset of

$$\mathscr{C}(\mathbb{R}^n) = \text{ the set of all non-empty compact subsets in } \mathbb{R}^n$$

on which we have defined the so-called Hausdorff metric [c.f. PSet 3-3-3]

$$d_H(A_1, A_2) := \inf\{r \mid A_1 \subset B(A_2, r) \text{ and } A_2 \subset B(A_1, r)\},$$

where $B(A,r) := \cup_{x \in A} B(x,r)$. So in particular, $\mathfrak{C}(\mathbb{R}^n)$ is a metric space.

We shall prove

Theorem 3.1 (Blaschke selection theorem)**.** *For any $R > 0$, the set of all nonempty compact convex subsets contained in $B(0, R)$ is compact (with respect to $\mathscr{T}_{d_H}$).*

As a consequence, any "bounded" sequence of compact convex sets has a subsequence which converges to a compact convex set with respect to d_H.

There are several different ways to prove Blaschke selection theorem. For example, one can prove that $\mathfrak{C}(\overline{B(0, R)})$ is a closed subset in $\mathscr{C}(\overline{B(0, R)})$. Thus the compactness follows from PSet 3-3-3. Here we give another proof via Arzela-Ascoli. To do so we list some standard results from convex geometry that we will not prove.

Definition 3.2. Let $A \subset \mathbb{R}^n$ be a convex compact subset. The *support function* of A is defined to be

$$h_A : \mathbb{R}^n \to \mathbb{R}, \quad h_A(v) = \sup_{x \in A} \langle x, v \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product.

It turns out that the support function characterize A:

Proposition 3.3. *For any compact convex set $A \subset \mathbb{R}^n$, the support function h_A is continuous, positive homogeneous and sub-additive:*

$$h_A(\alpha v) = \alpha h_A(v) \quad (\alpha > 0) \qquad \text{and} \qquad h_A(v_1 + v_2) \le h_A(v_1) + h_A(v_2).$$

Conversely for any continuous, positive homogeneous and sub-additive function h, there exists a unique compact convex domain A such that $h = h_A$.

As a result, the support function is a very important tool in convex geometry: it converts problems on geometric shapes to problems on continuous functions.

In fact, the Hausdorff distance between two compact sets can be computed via their support functions. Note that by positive homogeneity, each h_A is uniquely determined by its restriction

$$\tilde{h}_A = h_A|_{S^{n-1}} : S^{n-1} \to \mathbb{R}$$

which is a continuous function on S^{n-1}.

Proposition 3.4. *For any two compact sets A and B in $\mathbb{R}^n$,*

$$d_H(A, B) = d_u(\tilde{h}_A, \tilde{h}_B),$$

where d_u is the uniform metric on $\mathcal{C}(S^{n-1}, \mathbb{R})$.

We also need the following result:

Lemma 3.5. *Suppose $A \subset \overline{B(0, R)}$. Then for any $u, v \in \mathbb{R}^n$, we have*

$$|h_A(u) - h_A(v)| \le R|u - v|.$$

Now we can proceed to prove

Proof of Blaschke selection theorem.

By Lemma 3.5, the family of functions

$$\mathcal{F} = \{\tilde{h}_A \mid A \text{ is compact convex inside } \overline{B(0,R)}\} \subset \mathcal{C}(S^{n-1}, R)$$

is equicontinuous. Moreover, by definition it is pointwise bounded by R. It follows from Arzela-Ascoli theorem that any sequence of functions in $\mathcal{F}$ has a subsequence that converges uniformly to a continuous function $\tilde{h} \in \mathcal{C}(S^{n-1}, \mathbb{R})$. Define $h : \mathbb{R}^n \to \mathbb{R}$ to be the function that is positive homogeneous such that the restriction of h to S^{n-1} is $\tilde{h}$. Since $\tilde{h}$ is the uniform limit of a sequence of "restricted functions" whose "original function" are sub-additive, it is easy to see that h is also sub-additive. It follows that h is the support function of a compact convex set in $\overline{B(0,R)}$. So in view of Proposition 3.4, the theorem is proved. □

The Blaschke selection theorem has many applications. For example, one can prove isoperimetric inequality as follows.

Theorem 3.6 (Isoperimetric problem)**.** *Among all planar closed curve of fixed perimeter, the curve that enclosed maximal area is the circle.*

Sketch of proof. [3]

- Step 1: One can replace a non-convex curve by a convex curve of the same perimeter but encloses more area.
- Step 2: [Steiner] If the curve is not the circle, then one can find another convex curve which encloses more area.
- Step 3: The area functional is continuous. By Blaschke, there must exists a curve which enclosed maximal area

Conclusion: It has to be the circle! □

In a similar way Blaschke selection theorem guarantees the existence of a solution to the following problem:

Lebesgue's Universal Covering Problem. *What is the minimum area of a convex shape that can cover every planar set of diameter one?*

Unfortunately although we know the solution exists, we don't know the exact solution. This problem is still open. It was proved by Pal in 1920 that the area

$$a \leq 2 - \frac{2}{\sqrt{3}} = 0.84529\cdots.$$

After 100 years hard work, the current record is given by Gibbs in October 2018:

$$a \leq 0.84409359\cdots.$$

On the other direction, Brass and Sharifi showed in 2005 that $a \geq 0.832$.

[3] Steiner first gave a "proof" without step 3, and is criticized by Perron for the lack of compactness.