Topology (H) Lecture 16
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PATH, CONTINUOUS DEFORMATION AND HOMOTOPY

Last time we learned:

- Connectedness: many different characterizations of disconnected
  - Generalized Intermediate Value Theorem
  - closure, union (under conditions)
  - connectedness is productive (and divisible)
- Connected components: closed (but not necessarily open).
  - A numerical topological invariant: the number of connected components
  - $\pi_c : \mathcal{TOP} \to \mathcal{TOP}_{totdis}$

1. Path and Path-connectedness

We now turn to a closely related conception: the path connectedness. It is more intuitive, and, as we will see soon, can be extended to define "higher level" connectedness which is described by computable algebraic quantities.

Definition 1.1. Let X be a topological space, and $x, y \in X$. A *path* from x to y is a continuous map $\gamma : [0,1] \to X$ s.t.

$$\gamma(0) = x, \gamma(1) = y.$$

In the case $x = y$, we will call the path a *loop* with base point x.

Notations for *path space* and *loop space*:

$$\Omega(X; x_0, x_1) = \{\gamma \in \mathcal{C}([0,1], X) \mid \gamma(0) = x_0, \gamma(1) = x_1\},$$
$$\Omega(X; x_0) = \{\gamma \in \mathcal{C}([0,1], X) \mid \gamma(0) = \gamma(1) = x_0\}.$$

Remark. In our definition, path is a continuous map, not just a "geometric curve". In other words, different parametrizations of the same geometric curve will be regarded as different paths.

Remark. It is possible to define some "algebraic operations" on paths. For example,

- Given any path γ from x to y, we can "reverse" the path to get a new path $\bar{\gamma}$ from y to x by letting
$$\bar{\gamma}(t) := \gamma(1-t).$$
[The map $\bar{\gamma}$ is continuous because it is the composition of two continuous maps: the map γ and the map $t \mapsto 1-t$.]

- Given two paths, γ_1 from x to y and γ_2 from y to z, we can "connect" the two paths to get a new path $\gamma_1 * \gamma_2$ from x to z by letting

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t), & 0 \le t \le \frac{1}{2}, \\ \gamma_2(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

  [The continuity of $\gamma_1 * \gamma_2$ follows from the pasting lemma in PSet 2-2-2.]
- There is a special path from x to x: the constant path γ_x defined by

$$\gamma_x(t) = x, \quad \forall t \in [0,1].$$

Unfortunately these operations are not "very algebraic". For example, $\gamma * \bar{\gamma}$ is different from $\bar{\gamma} * \gamma$, since the first one is a path from x to x while the second one is a path from y to y. Even in the case $x = y$, they are still different paths since they are loops going in "opposite directions". Also we want the constant path γ_x to be the identity element, but it is not. We will show how to solve this problem and develop a correct "algebra of paths" next time.

Definition 1.2. We say a topological space X is *path-connected* is any two points in X can be connected by a path.

It is easy to prove that the conception of path-connectedness is stronger than connectedness:

Proposition 1.3. *If X is path-connected, then X is connected.*

Proof. By contradiction. Suppose there exists nonempty disjoint open sets A and B such that $X = A \cup B$. Take a point x in A, y in B and a path γ from x to y. Then

$$[0,1] = \gamma^{-1}(A) \cup \gamma^{-1}(B)$$

is the union of non-empty disjoint open sets, which contradicts with the connectedness of [0, 1]. □

Example.

(1) Any connected open set $U \subset \mathbb{R}^n$ is path connected.

Reason: *[The continuity method]* Fix any $x \in U$ and consider the set

$$A = \{y \in U \mid \text{ there exists a path from } x \text{ to } y\}.$$

Then

- A is open: For any $y \in A$, we take $\varepsilon > 0$ small enough such that $B(y, \varepsilon) \subset U$. Let γ_1 be a path in U connecting x to y. For any $y_1 \in B(y, \varepsilon)$, let γ_2 be the "line segment path" connecting the center y to y_1, which is given explicitly by

$$\gamma_1(t) = ty_1 + (1-t)y.$$

  Then $\gamma * \gamma_1$ is a path from x to y_1. So $y_1 \in A$.

- A is closed: By the same argument one can prove if $y \notin A$, then for any point $y_1 \in B(y, \varepsilon)$, we also have $y_1 \notin A$. So A^c is open, i.e. A is closed.

Since U is connected and since A is non-empty (we always have $x \in A$ since we have the constant curve), we conclude $A = U$. So any point in U can be connected to x. It follows that any two points can be connected by a path: first connect one point to x, then connect x to the other point.

By the same argument one can prove:

Fact: A topological manifold is path connected if and only if it is connected.

(2) $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected.

Reason: Since $\mathbb{Q}^2$ is a countable set, for any $x \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, there exists uncountably many lines l s.t.

$$x \in l \subset \mathbb{R}^2 \setminus \mathbb{Q}^2.$$

Now for $x \neq y \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, pick two such lines, one contains x and the other contains y, such that they are not parallel. Now you travel from x through the first line to the intersection point, then through the second line to y.

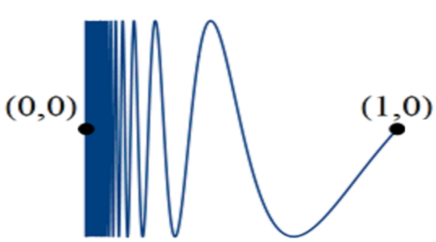
(3) The topologist's sine curve

$$X = \{(x, \sin\frac{\pi}{x}) \mid 0 < x \leq 1\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$$

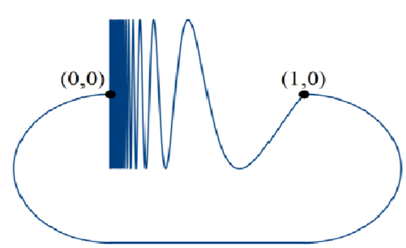
is connected (we have seen this in Lecture 15). But it is NOT path connected.

Reason: There is no path in X connecting the point $(0,0)$ to $(1,0)$. To see this we suppose $\gamma : [0,1] \to X$ is a path with $\gamma(0) = (0,0)$ and $\gamma(1) = (1,0)$. Write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Let $s = \sup\{t \mid \gamma_1(t) = 0\}$. Then $s < 1$, $\gamma_1(s) = 0$ and $\gamma_1(t) > 0$ for all $t > s$. It follows that for $t > s$, $\gamma_2(t) = \sin \frac{\pi}{\gamma_1(t)}$. Now take a decreasing sequence $t_n \to s$ with $\gamma_1(t_n) = \frac{2}{2n+1}$. [The existence of such sequence is guaranteed by the continuity of γ_1.] Then $\gamma_2(t_n)$ is an oscillating sequence and thus does not converge to $\gamma_2(s)$, a contradiction.

Of course you can make the space path connected by adding a path:



Not path-connected

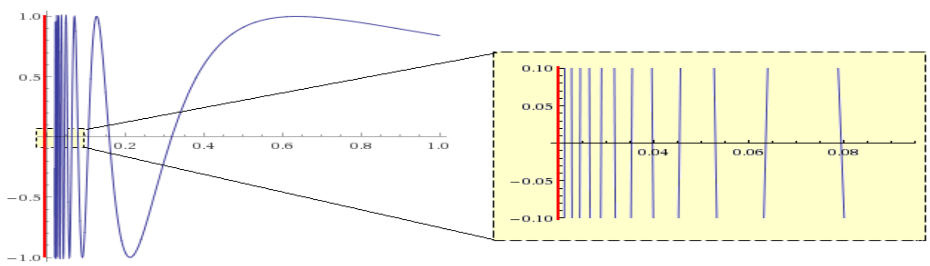


Path connected

Remark. So in general

- connected space need not be path connected.
- the closure of path connected subset need not be path connected.

If you think about the above example carefully, you will find that near any "bad point", say $(0,0)$, inside any small neighborhood you can find infinitely many "vertical curves" that are disconnected:



In other words, it is not "locally path connected" at these bad points:

Definition 1.4. We say a topological space X is *locally path connected at x* if for any open neighborhood U of x, there exists an open neighborhood V of x inside U which is path connected.

As usual, we say a topological space X is *locally path connected* if it is locally path connected at any point.

For example, any open set in $\mathbb{R}^n$ (or more generally any locally Euclidian space which includes all topological manifolds) is locally path connected. It turns out that any connected topological space without such bad points is path-connected, and the proof is the same as the proof of path connectedness for connected Euclidian domains above:

Proposition 1.5. *If X is connected and locally path connected, then X is path connected.*

Proof. Fix any $x \in X$. Let

$$P = \{y \in X \mid y \text{ can be connected by path to } x\}.$$

By locally path connectedness,

- if a point is in P, then a neighborhood of this point is in P,
- if a point is in P^c, then a neighborhood of this point is in P^c.

So P is both open and closed. Since X is connected and P is non-empty, we must have $P = X$. □

Last time we showed connectedness is preserved under continuous maps, under union with nonempty intersection, and under products. The same properties holds for path connectedness, and the proofs are simpler:

Proposition 1.6. *Let $f : X \to Y$ be continuous. Then for any path connected subset $A \subset X$, the image $f(A)$ is path connected.*

Proof. For any $f(x_1), f(x_2) \in f(A)$, we pick a path $\gamma : [0,1] \to A$ from x_1 to x_2. Then $f \circ \gamma : [0,1] \to f(A)$ is a path from $f(x_1)$ to $f(x_2)$. □

As a consequence, any quotient space of a path connected space is still path connected. So like connectedness, "path connectedness" is also a divisible topological property.[1]

Proposition 1.7. *Let X_α be path connected and $\cap_\alpha X_\alpha \neq \emptyset$. Then $\cup_\alpha X_\alpha$ is path connected.*

Proof. Take $x_0 \in \cap_\alpha X_\alpha$. For any $x_1 \in X_{\alpha_1}$ and $x_2 \in X_{\alpha_2}$, there exist paths γ_1 from x_1 to x_0 and γ_2 from x_0 to x_2. It follows $\gamma_1 * \gamma_2$ is a path from x_1 to x_2. □

Proposition 1.8. *If each X_α is path connected, then the product space $\prod_\alpha X_\alpha$ is also path connected (w.r.t. the product topology).*

Proof. For any $(x_\alpha), (y_\alpha) \in \prod_\alpha X_\alpha$, we pick paths $\gamma_\alpha : [0,1] \to X_\alpha$ from x_α to y_α. Then

$$\gamma : [0,1] \to \prod_\alpha X_\alpha, \quad \gamma(t) = (\gamma_\alpha(t))$$

is continuous and is a path from $\gamma(0) = (x_\alpha)$ to $\gamma(1) = (y_\alpha)$. [Here we used the fact a map to the product is continuous iff each component of the map is continuous.] □

We can also define an equivalence relation via path:

$$x \overset{p}{\sim} y \iff \exists \text{ path in } X \text{ connecting } x \text{ and } y.$$

It is easy to check $\overset{p}{\sim}$ is a equivalence relation: the three conditions for equivalence relation are exactly the three items in the remark at the end of page 1.

Definition 1.9. Let X be a topological space.

(1) Each equivalence class for $\overset{p}{\sim}$ is called a *path component* of X.
(2) The set of $\overset{p}{\sim}$-equivalence classes is denoted by $\pi_0(X)$.

Remark. Last time we proved that any connected component is a closed subset (which need not be open). From the example of topologist's sine curve we conclude that a path component could be neither closed nor open.

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[1]A topological property P is called *divisible* if for any X satisfying (P), any quotient space of X also satisfies (P).

Since any continuous map $f : X \to Y$ will map a path component in X into a path component in Y, we naturally get a well-defined map

$$\pi_0(f) : \pi_0(X) \to \pi_0(Y), \qquad [x] \mapsto [f(x)].$$

Obviously in the case f is a homeomorphism, the map $\pi_0(f)$ is a bijection whose inverse is given by $\pi_0(f^{-1})$.

Again, we can think of $\pi_0(X)$ as a quotient topological space (with the quotient topology). For the case of topologist's sine curve, the quotient space $\pi_0(X)$ consists of two elements. Let's use "v" to represent the vertical line segment part and use "s" to represent the sine curve part. Then we have

$$\pi_0(\text{topologist's sine curve}) = \{v, s\}, \qquad \mathscr{T}_{quotient} = \{\emptyset, s, \{v, s\}\}.$$

This is a very bad topological space: it is not totally disconnected (in fact it is path connected via $\gamma(t) = s$ for $t < 1/2$ and $\gamma(t) = v$ for $t \geq 1/2$); AND it is not even (T1). [One can check that the space of connected components, $\pi_c(X)$, is always a (T1) space when endowed with the quotient topology.]

So although we can regard π_0 as a functor mapping topological spaces X to the quotient topological space $\pi_0(X)$ and mapping continuous functions $f : X \to Y$ to $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ [which *is* continuous w.r.t. the quotient topology], in general people would rather forget about the topological structure on the quotient since in general it will not give us any useful information.

In other words, we will only regard $\pi_0(X)$ as a set. It is easy to check that the maps $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ associated to continuous maps $f : X \to Y$ still satisfies the conditions of a functor. So from path components relation we get a functor

$$\pi_0 : \mathcal{TOP} \to \mathcal{SET}, \qquad X \mapsto \pi_0(X), f \mapsto \pi_0(f).$$

Remark. Of course we lost a lot of information when applying the functor π_0. But this is exactly the philosophy of algebraic topology: to distinguish topological spaces could be very hard, but very often it would be easier to distinguish objects in simpler categories (like $\mathcal{SET}$, $\mathcal{GROUP}$, or $\mathcal{VECTORSPACE}$). For example, by counting the cardinality of $\pi_0(X)$, we are able to distinguish many topological spaces, e.g. in Lecture 1 we have mentioned

$$3 \neq 4 \qquad \text{THREE} \neq \text{FOUR}$$

The second group of figures are topologically different because they have different π_0. For the first group, one can either use π_0 by carefully deleting points in each sides, or by looking at π_1, the fundamental group, which counts the "holes" in the figure.

2. Continuous deformation

In Lecture 1 we have seen the importance of "continuous deformation" in topology via pictures, but without giving it a precise definition. With general topology at hand, we can always give a precise meaning when we talk about "continuous" objects in abstract setting:

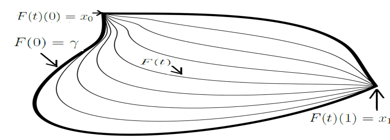
> A *continuous deformation* of an object x_0 in an abstract topological space X is a continuous map $f : [0,1] \to X$ with $f(0) = x_0$, which may have some extra constraints depending on the problem.

For example, given a path γ from x_0 to x_1 inside a space X, a *continuous deformation of the path* γ with endpoints fixed is a continuous map

$$F : [0,1] \to \Omega(X; x_0, x_1) = \{\gamma \in \mathcal{C}([0,1], X) \mid \gamma(0) = x_0, \gamma(1) = x_1\}$$

which satisfies conditions

$$F(0) = \gamma,$$
$$F(t)(0) = x_0,$$
$$F(t)(1) = x_1.$$



Wait a minute! We have not specify a topology on the path space $\Omega(X; x_0, x_1)$ yet. Without a topology, it would make no sense to talk about the continuity of F!

Fortunately $\Omega(X; x_0, x_1)$ is a subspace of $\mathcal{C}([0,1], X)$, on which we already have several topologies. In the general case where X is a topological space, from the "convergence point of view" the best topology on $\mathcal{C}([0,1], X)$ is the compact-open topology. [Note: we don't want to use the product (=pointwise convergence) topology here, since we want the limit of a sequence of path (=continuous maps) to be a path!]

More generally, consider a map $f \in \mathcal{C}(X, Y)$, where X, Y are topological spaces. A *continuous deformation of* f over a parameter space T should be a continuous map

$$F : T \to \mathcal{C}(X, Y), \qquad t \mapsto F(t) = f_t \in \mathcal{C}(X, Y)$$

such that $f_{t_0} = f$ for some $t_0 \in T$, where the topology on $\mathcal{C}(X, Y)$ is the *compact-open topology* $\mathscr{T}_{c.o.}$. By definition, this topology is generated by a sub-base

$$\mathscr{S}_{c.o.} = \{S(K, U) \mid K \subset X \text{ is compact and } U \subset Y \text{ is open } \},$$

where

$$S(K, U) = \{f \in \mathcal{C}(X, Y) \mid f(K) \subset U\}.$$

Now suppose we have a continuous family (with parameter space T) of maps in $\mathcal{C}(X, Y)$. That is , we have a map $F \in \mathcal{C}(T, \mathcal{C}(X, Y))$. This is still conceptionally complicated. But given any such F we can define a much simpler map

$$G \in \mathcal{M}(T \times X, Y), \quad G(t, x) := F(t)(x).$$

It turns out that under mild conditions, F is continuous if and only if G is continuous!

Proposition 2.1. *Suppose X is locally compact Hausdorff, Y, T are arbitrary topological spaces. Consider the correspondence (which is bijection)*

$$\mathcal{M}(T, \mathcal{M}(X, Y)) \longleftrightarrow \mathcal{M}(T \times X, Y)$$
$$F(t)(x) \longleftrightarrow G(t, x) := F(t)(x)$$

Then $F \in \mathcal{C}(T, \mathcal{C}(X, Y))$ if and only if $G \in \mathcal{C}(T \times X, Y)$.

Proof. ($\Longleftarrow$) Suppose $G \in \mathcal{C}(T \times X, Y)$. Then given any $t \in T$, $F(t)$ is continuous since it can be written as the composition of continuous maps

$$X \xrightarrow{j_t} T \times X \xrightarrow{G} Y,$$

where $j_t(x) = (t, x)$ is the "canonical embedding at level t". So F maps T into $\mathcal{C}(X, Y)$. To prove F is continuous as a map from T to $\mathcal{C}(X, Y)$, it is enough to prove

(*) $\quad F^{-1}(S(K, U))$ is open in T for any compact $K \subset X$ and open $U \subset Y$.

Suppose $F(t) \in S(K, U)$. Then by definition of G, $G(\{t\} \times K) \subset U$, i.e.

$$\{t\} \times K \subset G^{-1}(U).$$

By continuity of G, $G^{-1}(U)$ is open in $T \times X$. Since K is compact, by the tube lemma in Lecture 6, there exists open sets $V \ni t$ in T and $W \supset K$ in X such that

$$V \times W \subset G^{-1}(U).$$

It follows that for any $t \in V$,

$$F(t)(K) \subset G(V \times K) \subset G(V \times W) \subset U.$$

It follows that $V \subset F^{-1}(S(K, U))$, which proves (*).

($\Longrightarrow$) Suppose $F \in \mathcal{C}(T, \mathcal{C}(X, Y))$. To prove G is continuous, we need to show

(**) $\quad G^{-1}(U)$ is open in $T \times X$ for any open set $U \subset Y$.

Suppose $G(t, x) \in U$, i.e. $F(t)(x) \in U$. Then $F(t) \in S(\{x\}, U)$. In PSet 5-3-1 we have seen that if X is locally compact and Hausdorff, then

$$S(\{x\}, U) = \bigcup_{\text{compact neighborhood } K \text{ of } x} S(K, U).$$

So there exists an open neighbourhood W_x of x s.t. $\overline{W_x}$ is compact, and

$$F(t) \in S(\overline{W_x}, U).$$

Since F is continuous, and $t \in F^{-1}(S(\overline{W_x}, U))$, there must exists an open neighborhood V of t s.t.

$$V \subset F^{-1}(S(\overline{W_x}, U)),$$

i.e. $G(V, \overline{W_x}) \subset U$. It follows

$$V \times W_x \subset V \times \overline{W_x} \subset G^{-1}(U).$$

This completes the proof. $\square$

Remark. We only used LCH in the proof of the second part. In other words, without LCH condition we can still claim: Any function $G \in \mathcal{C}(T \times X, Y)$ defines a continuous family of continuous maps: $F \in \mathcal{C}(T, \mathcal{C}(X, Y))$.

3. Homotopy of maps

Now we introduce the most important object on which we study the path and path components. Suppose X and Y are topological spaces and $f_0, f_1 \in \mathcal{C}(X, Y)$.

Definition 3.1. We say $f_0, f_1 \in \mathcal{C}(X, Y)$ are *homotopic* if there is a continuous map

$$F : [0,1] \times X \to Y$$

such that $F(0, x) = f_0(x), F(1, x) = f_1(x), \forall x \in X$. Such an F is called a *homotopy* between f_0 and f_1.

Notation: If f_0 is homotopic to f_1, we will write

$$f_0 \sim f_1.$$

In view of proposition 2.1, when X is LCH we have

$$\begin{aligned} & f_0 \text{ and } f_1 \text{ are homotopic} \\ \Longleftrightarrow & f_0 \text{ can be deformed continuously to } f_1 \text{ with parameter space } [0,\ 1] \\ \Longleftrightarrow & f_0 \text{ and } f_1 \text{ lie in the same path component in } (\mathcal{C}(X, Y), \mathscr{T}_{c.o.}). \end{aligned}$$

In view of the remark above, even in the bad case where X is not LCH, we are still safe to say: if f_0 and f_1 are homotopic, then they lie in the same path component of $(\mathcal{C}(X, Y), \mathscr{T}_{c.o.})$.

It is easy to check that homotopy gives an equivalence relation on $\mathcal{C}(X, Y)$:

- $f \sim f$,
- $f_1 \sim f_2 \Longrightarrow f_2 \sim f_1$,
- $f_1 \sim f_2, f_2 \sim f_3 \Longrightarrow f_1 \sim f_3$.

In the case where X is LCH, this is just the equivalence relation defined by path in $\mathcal{C}(X, Y)$.

Notation: For each $f \in \mathcal{C}(X, Y)$, we denote

$$[f] = \text{the homotopy equivalence class containing } f$$

and we denote

$$[X, Y] = \mathcal{C}(X, Y)/\!\sim\; = \text{the set of homotopy classes.}$$

So if X is LCH, then $[X, Y] = \pi_0(\mathcal{C}(X, Y))$.

A special case: If $X = \{\text{pt}\}$ is a single point set, then a continuous map $f : X \to Y$ is equivalent to a point in Y. In this case "two continuous maps f_0, f_1 are homotopic" is equivalent to "two points can be connected by a path in Y". In other words,

$$[\{pt\}, Y] = \pi_0(Y).$$

Here are some natural operations on homotopy classes of maps

(1) Composition

$$[X, Y] \times [Y, Z] \to [X, Z]$$
$$([f], [g]) \mapsto [g \circ f].$$

(2) Pull-back

$$F : X_0 \to X_1 \quad \rightsquigarrow \quad F^* : [X_1, Y] \to [X_0, Y]$$
$$[f] \mapsto F^*([f]) = [f \circ F].$$

(3) Push-forward

$$F : Y_0 \to Y_1 \quad \rightsquigarrow \quad F_* : [X, Y_0] \to [X, Y_1]$$
$$[f] \mapsto F_*([f]) = [F \circ f].$$

One can check that these operations are well-defined, i.e. they are independent of the choices of representatives in each class.

The simplest continuous maps are the constant maps, i.e. mapping all points in X to a single point in Y.

Definition 3.2. $f \in \mathcal{C}(X, Y)$ is *null-homotopic* if it is homotopic to a constant map.

The conception is very useful in geometry.

Example.

(1) Let $Y \subset \mathbb{R}^n$ be convex, or more generally, be a "star-shaped" region, i.e.

$$\exists y_0 \in Y \text{ such that } \forall y \in Y, \text{ the line segment } \overline{y_0 y} \subset Y.$$

Then for any X,

- any map $f \in \mathcal{C}(X, Y)$ is null-homotopic.
  Reason: The homotopy from f to a constant map is given by
  $$F : [0, 1] \times X \to Y, \quad t \mapsto F(t, x) = ty_0 + (1 - t)f(x).$$
- any map $f \in \mathcal{C}(Y, X)$ is null-homotopic.
  Reason: The homotopy from f to a constant map is given by
  $$F : [0, 1] \times Y \to X, \quad t \mapsto F(t, y) = f(ty_0 + (1 - t)y).$$

Definition 3.3. We say a topological space X is *contractible* if the identity map Id_X is null-homotopic.

Example: Any star-shaped region in $\mathbb{R}^n$ is contractible.

(2) Let $X = Y = S^1 \subset \mathbb{C}$. Let $f_n \in \mathcal{C}(X, Y)$ be the map
$$f_n(z) = z^n.$$
We will see that all these f_n are NOT homotopic to each other, and are not null-homotopic.

(3) Let i be the inclusion map
$$i : S^{n-1} \hookrightarrow B^n = \{x \in \mathbb{R}^n \Big| |x| \leqslant 1\}.$$
<u>Fact</u>: There exists $f \in \mathcal{C}(B^n, S^{n-1})$ with
$$f \circ i = \mathrm{Id}_{S^{n-1}}$$
if and only if $\mathrm{Id}_{S^{n-1}} \in \mathcal{C}(S^{n-1}, S^{n-1})$ is null-homotopic.

Proof. • If such an f exists, then $\mathrm{Id}_{S^{n-1}} \sim c$ via
$$F : [0,1] \times S^{n-1} \to S^{n-1}, (t, x) \mapsto f(tx),$$
where
$$F(0, x) = f(0) \in S^{n-1}$$
is a constant point, and
$$F(1, x) = f(x) = x$$
on S^{n-1}.

• Conversely if there exists $F : [0,1] \times S^{n-1} \to S^{n-1}$ s.t.
$$F(0, x) = c, \quad F(1, x) = x.$$
Then we can define $f : B^n \to S^{n-1}$ by
$$f(x) = \begin{cases} F(|x|, \frac{x}{|x|}) & x \neq 0, \\ c & x = 0. \end{cases}$$
It remains to check f is continuous at $x = 0$. Since S^{n-1} is compact, the continuous function F is uniformly continuous. So $\forall \varepsilon > 0, \exists \delta > 0$ s.t.
$$|F(t, x) - F(0, x)| < \varepsilon$$
for $\forall t < \delta$ and $\forall x \in S^{n-1}$. So f is continuous. □

Note: We will see that for any n, $\mathrm{Id}_{S^{n-1}}$ is never null-homotopic. (This is equivalent to Brouwer's fixed point theorem: Any continuous function $f : B^n \to B^n$ has a fixed point!)