Topology (H) Lecture 22
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BROUWER'S FIXED POINT THEOREM AND INVARIANCE OF DOMAIN

Last time:
Let X be path-connected, locally path-connected and semi-locally simply connected.

- X admits a universal covering space $\widetilde{X}$.
- There is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$.
- There is a bijection between isomorphism classes of path-connected covering spaces $p : \widetilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

Today: Brouwer's fixed point theorem, the invariance of domain theorem.

1. Brouwer's fixed point theorem

We proved in Lecture 18 that any continuous map $f : \overline{D} \to \overline{D}$ has a fixed point, where $\overline{D}$ is the closed disk in $\mathbb{R}^n$. Today we want to prove its higher dimensional analogue. Denote $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ and $\overline{B^n} = \{x \in \mathbb{R}^n : |x| \leqslant 1\}$.

Theorem 1.1 (Brouwer's fixed point theorem)**.** [1]
For any continuous map $f : \overline{B^n} \to \overline{B^n}$, there exists $p \in \overline{B^n}$ s.t. $f(p) = p$.

As a consequence, $\pi_n(S^n) \neq \{e\}$.

We will follow the proof given by J. Milnor (which was simplified by Rogers). The proof is based on "differential topology". (There are also proofs based on "algebraic topology" and proofs based on "combinatorial topology".) The idea is:

- First prove a "simpler" version where the map under consideration is smooth.
- Reduce the continuous case to the smooth case by using approximation.

Usually it is much easier to prove a theorem for smooth maps than for continuous maps, because we have a very powerful tool: the *differential* (or in other words, *linear approximation*).

[1] It is said that Brouwer discovered this theorem while stirring a cup of coffee and noticing that there is always at least one point in the liquid that does not move. Here is some true history: The case $n = 3$ was first proven by P. Bohl in 1904, and then by Brouwer in 1909. The general case was first proven by Hadamard in 1910, and then by Brouwer in the same year via a different method. The revolutionary aspect of Brouwer's approach was his systematic use of algebraic topology.

Recall that if $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ are open sets and

$$f = (f_1, \cdots, f_m) : U \to V$$

is a C^1 map, then at each point $x \in U$, the "differential" $\mathrm{d}f_x$ is a linear map

$$\mathrm{d}f_x = \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} : \mathbb{R}^n \to \mathbb{R}^m,$$

$$\vec{v} \mapsto \mathrm{d}f_x(\vec{v}) = \left(\frac{\partial f_i}{\partial x_j}\right)\vec{v} = \left(\sum_j \frac{\partial f_1}{\partial x_j}v_j, \cdots, \sum_j \frac{\partial f_m}{\partial x_j}v_j\right)^T.$$

In mathematical analysis we have seen

- $\mathrm{d}(f \circ g)_x = \mathrm{d}f_{g(x)} \circ \mathrm{d}g_x$,
- $\mathrm{d}(\mathrm{Id}_U)_x = \mathrm{Id}_{\mathbb{R}^n}$.

[So the differential d is a functor from the category "Euclidian domains with morphism being C^1 maps" to the category "linear spaces with morphism being linear maps".]

To illustrate the use of differential, we first prove

Theorem 1.2 (Invariance of dimension theorem, smooth version)**.** *Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be open. If $m \neq n$, then there does not exists a C^1 diffeomorphism*[2] *$f : U \to V$.*

Proof. Suppose $f : U \to V$ is a C^1 morphism. by taking differential we get two linear maps, $\mathrm{d}f_x : \mathbb{R}^n \to \mathbb{R}^m$ and $(\mathrm{d}f^{-1})_{f(x)} : \mathbb{R}^m \to \mathbb{R}^n$. Moreover,

$$f \circ f^{-1} = \mathrm{Id}_V \Longrightarrow \mathrm{d}f_x \circ \mathrm{d}f^{-1}_{f(x)} = \mathrm{Id}_{\mathbb{R}^m}$$
$$f^{-1} \circ f = \mathrm{Id}_U \Longrightarrow \mathrm{d}f^{-1}_{f(x)} \circ \mathrm{d}f_x = \mathrm{Id}_{\mathbb{R}^n}.$$

So the linear maps $\mathrm{d}f_x$ and $\mathrm{d}f^{-1}_{f(x)}$ are inverse to each other, which implies $m = n$. □

Back to the proof of Brouwer's fixed point theorem. Before we give the proof, let's first go over the proof for $n = 2$. We showed that if $f(p) \neq p, \forall p \in \overline{D}$. Then there is a retraction $g : \overline{D} \to S^1$ of the form

$$g(p) = p + \lambda(p)(p - f(p)).$$

If you do the computation carefully, you will get

$$\lambda(p) = \frac{-p \cdot (p - f(p)) + [(p \cdot (p - f(p)))^2 + |p - f(p)|^2(1 - |p|^2)]^{\frac{1}{2}}}{|p - f(p)|^2}.$$

Key observations:

- The terms in $[\cdots]^{\frac{1}{2}}$ is positive. As a consequence, the map $g : \overline{D} \to S^1$ is C^1.
- The construction of g works for all dimension (with the same formula for $\lambda(p)$).

[2] A map $f : U \to V$ is called a C^1 *diffeomorphism* if f is C^1, bijective and f^{-1} is also C^1.

To prove Brouwer's fixed point theorem, we first prove the following smooth analogue of "no retraction theorem":

Theorem 1.3 (No smooth retraction)**.**
There exists no C^1 *map* $f : \overline{B^n} \to S^{n-1}$ *s.t.* $f|_{S^{n-1}} = \mathrm{Id}$.

"No smooth retraction" implies "Brouwer's fixed point theorem".

Let $f : \overline{B^n} \to \overline{B^n}$ be continuous. By Stone-Weierstrass theorem, for each $l \in \mathbb{N}$ there exists a smooth map (in fact a "polynomial") $p_l : \overline{B^n} \to \mathbb{R}^n$ s.t.

$$|p_l(x) - f(x)| < \frac{1}{l}, \quad \forall x \in \overline{B^n}.$$

The image of p_l could be outside the ball. But we can "shrink" it by defining

$$f_l := \frac{l}{l+1} p_l.$$

Then $f_l : \overline{B^n} \to \overline{B^n}$ is C^1, and $f_l \to f$ uniformly on $\overline{B^n}$.

If f_l has no fixed point, then we define $g_l : \overline{B^n} \to S^{n-1}$ as in the 2-dim case. Then g_l is C^1, and $g_l|_{S^{n-1}} = \mathrm{Id}$, which contradicts with the no smooth retraction theorem. So for any l, there exists $x_l \in \overline{B^n}$ s.t. $f_l(x_l) = x_l$. Picking a convergent subsequence $x_{l_i} \to x_0$, we get

$$f(x_0) = \lim_{i\to\infty} f_{l_i}(x_{l_i}) = \lim_{i\to\infty} x_{l_i} = x_0.$$

This completes the proof of the Brouwer's fixed point theorem. □

Proof of "No smooth retraction" theorem.

> Idea: Suppose there is a smooth retraction. Since $\overline{B^n}$ is contractible, we can find a C^1-homotopy connecting the identity to the retraction. Then by taking differential we will get a continuous family of linear maps from the identity linear map to the differential of the retraction which is a singular map. To get a contradiction, we can try to find a numerical quantity describing this change of differentials and work on it.

Suppose there exists C^1 map $f : \overline{B^n} \to S^{n-1}$ s.t.

$$f|_{S^{n-1}} = \mathrm{Id}.$$

For $t \in [0,1]$, we let $f_t : \overline{B^n} \to \overline{B^n}$ be the homotopy connecting the identity map $\mathrm{Id}_{\overline{B^n}}$ and f (as a map into $\overline{B^n}$), i.e.

$$f_t(x) = (1-t)x + tf(x) = x + t(f(x) - x) =: x + tg(x).$$

To study the differentials along the homotopy, we define a map $F : [0,1] \to \mathbb{R}$ by

$$F(t) = \int_{\overline{B^n}} \det(\mathrm{d}f_t)_x \, \mathrm{d}x = \int_{\overline{B^n}} \det(I + t\mathrm{d}g_x) \, \mathrm{d}x.$$

By linear algebra, F is a polynomial in t, for $t \in [0,1]$.

On one hand, for $t = 1$ we have

$$f_1(x) = f(x) \in S^{n-1}, \quad \forall x \in \overline{B^n}.$$

So for any $x \in B^n$ and any $\vec{v}$ in a small ball with $x + t\vec{v} \in \overline{B^n}$, we have

$$2\langle (\mathrm{d}f_1)_x \vec{v}, f_1(x)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \langle f_1(x+t\vec{v}), f_1(x+t\vec{v})\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} 1 = 0.$$

As a consequence,

$$\mathrm{Im}((\mathrm{d}f_1)_x) \subset f_1(x)^{\perp} = \{u \in \mathbb{R}^n \mid u \perp f_1(x)\}.$$

This implies $\mathrm{rank}((\mathrm{d}f_1)_x) \leqslant n-1$ and thus $\det((\mathrm{d}f_1)_x) = 0, \forall x \in B^n$. So

$$F(1) = \int_{\overline{B^n}} \det(\mathrm{d}f_1)_x \, \mathrm{d}x = 0,$$

On the other hand, since $f_0|_{\overline{B^n}} = \mathrm{Id}$, one may reasonably guess

Claim: $\exists t_0 > 0$ s.t. $f_t : \overline{B^n} \to \overline{B^n}$ is a C^1-diffeomorphism for $0 \leqslant t \leqslant t_0$.

Let's first assume the claim is true. By the change of variable formula,

$$F(t) = \int_{f_t(\overline{B^n})} \mathrm{d}x = \mathrm{Vol}(\overline{B^n}), \quad \forall t \in [0, t_0].$$

So F is a polynomial which equals to a constant in a small interval. It follows that F must equal to the constant for all t (use the analyticity of polynomial [Lecture 15] or the fundamental theorem of algebra). In particular,

$$F(1) = \mathrm{Vol}(\overline{B^n}) > 0,$$

a contradiction. □

It remains to prove the claim:

Proof of the claim.

Recall that $f_t(x) = x + tg(x)$, where $g(x) = f(x) - x$.

We first prove that there exists $t_1 > 0$ such that f_t is injective for $t \in [0, t_1]$. Suppose there exist $x_1 \neq x_2$, s.t. $f_t(x_1) = f_t(x_2)$. It follows

$$|x_1 - x_2| = t|g(x_1) - g(x_2)|.$$

Since f is C^1, g must be C^1. So there exists $C > 0$ s.t.

$$|g(x_1) - g(x_2)| \leqslant C|x_1 - x_2|, \quad \forall x_1, x_2 \in \overline{B^n}.$$

It follows that

$$|x_1 - x_2| \leqslant Ct|x_1 - x_2|.$$

This can't happen for $t < t_1 := \frac{1}{C}$, since $x_1 \neq x_2$. In other words, we proved: If $t < t_1$, then f_t is injective.

Then we prove there exists $t_0 > 0$ such that f_t is surjective for $t \in [0, t_0]$. Denote

$$G_t = f_t(B^n).$$

Since $f_t|_{S^{n-1}} = \mathrm{Id}$, we only need to show that $G_t = B^n$ for $t < t_0$. Since $\det(\mathrm{d}f_t)$ is continuous on t, and $\det(\mathrm{d}f_0) = 1$, there exists $t_2 > 0$ such that

$$\det(\mathrm{d}f_t) > 0, \quad \forall t \in [0, t_2].$$

By the inverse function theorem, f_t has a C^1-inverse which maps G_t to B^n for all $t \in [0, t_0]$, where $t_0 := \min\{t_1, t_2\}$. As a result, G_t is open for $t < t_0$. We want to show $G_t = B^n$ for $t < t_0$. Again by contradiction, suppose $G_t \neq B^n$. Since f_t is injective, there exists $y_0 \in \partial G_t \cap B^n$. Pick $x_l \in B^n$ s.t. $f_t(x_l) \to y_0$. By compactness of $\overline{B^n}$, there exists a subsequence

$$x_{l_i} \to x_0 \in \overline{B^n}.$$

By continuity of f_t, $f_t(x_0) = y_0$. Since G_t is open, $y_0 \notin G_t$. Then we must have $x_0 \notin B^n$, i.e. $x_0 \in S^{n-1}$. So we have

$$y_0 = f_t(x_0) = x_0 \in S^{n-1}.$$

This contradicts with the fact $y_0 \in B^n$.

Since we already proved f_t is bijective and has a C^1 inverse for $t \in [0, t_0]$, it is a C^1 homeomorphism for any $t \in [0, t_0]$. □

Brouwer's fixed point theorem is useful in many other subjects, in particular in economics. The following version of Brouwer's fixed point theorem is widely used:

Theorem 1.4 (Brouwer's fixed point theorem, 2^{nd} form)**.** *Let $K \subset \mathbb{R}^n$ be non-empty, compact and convex. Then any continuous map $f : K \to K$ has a fixed point.*

Sketch of proof (Details left as an exercise). By translation we may assume $0 \in K$. Let $V = \mathrm{span}_{\mathbb{R}} K$. Then V is a subspace of $\mathbb{R}^n$, and thus $V \simeq \mathbb{R}^m$ for some m. Moreover, $K \subset V$ and K is not in any proper hyperplane, thus K has non-empty interior. [This fact need a proof.] Hence convexity of K implies $K \simeq \overline{B^m}$. □

Brouwer's fixed point theorem has many generalizations. A natural question is: is it true for infinitely dimensional spaces?

Example. Consider the l^2-space

$$X = l^2 = \{(a_1, a_2, \cdots) | \sum_{i=1}^{\infty} a_i^2 < +\infty\}.$$

In Lecture 2 we have seen that X is a metric space with metric

$$d((a_i), (b_i)) = \sqrt{\sum_{i=1}^{\infty} (a_i - b_i)^2}.$$

Denote $\overline{B} = \overline{B(0, 1)}$. Consider the map

$$f : \overline{B} \to \overline{B},$$
$$a = (a_1, a_2, \cdots) \mapsto (\sqrt{1 - \|a\|_2^2}, a_1, a_2, \cdots).$$

Then f is continuous, since as $d(a,b) \to 0$ we have

$$[d(f(a), f(b))]^2 = (\sqrt{1-\|a\|_2^2} - \sqrt{1-\|b\|_2^2})^2 + \|a-b\|_2^2 \to 0.$$

However, f has no fixed point: If $f(a) = a$, then

$$a_1 = a_2 = \cdots = \sqrt{1-\|a\|_2^2},$$

which has no solution.

So the original form of Brouwer's fixed point theorem fails for infinitely dimensional spaces. One reason is: the closed unit ball $\overline{B}$ in l^2 is NOT compact. It turns out that the 2^{nd} form of Brouwer's fixed point theorem (where we work on compact convex sets instead of on closed unit balls) works in infinitely dimensional spaces:

Theorem 1.5 (Schauder's fixed point theorem)**.** *Let $\emptyset \neq K$ be a compact convex subset of a normed vector space V. Then any continuous map $f : K \to K$ has a fixed point.*

Proof. Since K is compact in a metric space, for any $\varepsilon > 0$ one can find a finite ε-net $\{x_1, \cdots, x_n\}$ for K. For $x \in K$ and $1 \leqslant i \leqslant n$ we define

$$\rho_i(x) = \begin{cases} 0, & d(x, x_i) > \varepsilon, \\ \varepsilon - d(x, x_i), & d(x, x_i) \leqslant \varepsilon. \end{cases}$$

Obviously each ρ_i is continuous, and for any $x \in K$, there exists i s.t. $\rho_i(x) > 0$. It follows that the map

$$\rho_\varepsilon : K \to K, \quad x \mapsto \frac{\sum_{i=1}^n \rho_i(x) x_i}{\sum_{i=1}^n \rho_i(x)}$$

is well-defined (here we used the convexity of K) and continuous, and satisfies

$$d(\rho_\varepsilon(x), x) < \varepsilon, \quad \forall x \in K$$

since $\rho_\varepsilon(x)$ is a convex combination of those x_i's that lie in $B(x, \varepsilon)$. Now for each n, consider the finite dimensional vector space

$$V_n = \text{span}\{x_1, \cdots, x_n\} \subset V.$$

Denote

$$K_\varepsilon = \text{conv}(x_1, \cdots, x_n)$$

be the (closed) convex hull of $x_1, \cdots, x_n$. Then $K_\varepsilon \subset K$ is homeomorphic to some Euclidean closed ball, and $f_\varepsilon = \rho_\varepsilon \circ f : K_\varepsilon \to K_\varepsilon$ is continuous. So there exists $x_\varepsilon \in K_\varepsilon \subset K$ s.t. $f_\varepsilon(x_\varepsilon) = x_\varepsilon$. This implies

$$d(x_\varepsilon, f(x_\varepsilon)) = d(f_\varepsilon(x_\varepsilon), f(x_\varepsilon)) = d(\rho_\varepsilon(f(x_\varepsilon)), f(x_\varepsilon)) < \varepsilon$$

Hence

$$\inf\{d(x, f(x)) | x \in K\} = 0.$$

Since K is compact, the minimum is achieved. □

2. Invariance of domain

As an application of Brouwer's fixed point theorem, we now drop the "smoothness" assumption in the Invariance of Dimension theorem, i.e. Theorem 1.2, to get a "topological" version:

Theorem 2.1 (Topological Invariance of Dimension)**.** *Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be open. If $m \neq n$, then $U \not\simeq V$.*

Although looks quit obvious, the proof is not easy. The theorem was first proven by Brouwer. In fact, what he proved is the following version.

Theorem 2.2 (Brouwer's Invariance of Domain)**.** *If $U \subset \mathbb{R}^n$ is open, and $f : U \to \mathbb{R}^n$ is injective continuous, then $f(U)$ is open in $\mathbb{R}^n$.*

Remark. The theorem fails for $n = +\infty$. For example, if we take

$$f : l^2 \to l^2, \quad (a_1, a_2, \cdots) \mapsto (0, a_1, a_2, \cdots),$$

then f is injective and continuous, but the image $f(l^2)$ is NOT open in l^2.

Brouwer's Invariance of Domain implies Topological Invariance of Dimension.

Let's assume that there exists a homeomorphism $f : U \to V$. We may assume $n > m$ (if $n < m$, then we use f^{-1} instead). Let $i : \mathbb{R}^m \to \mathbb{R}^n$ be the inclusion map

$$i(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0).$$

Then the composition

$$i \circ f : U \to \mathbb{R}^m \hookrightarrow \mathbb{R}^n$$

is injective and continuous, but the image is NOT open since it is contained in a lower dimensional subspace. This contradicts with Brouwer's Invariance of Domain theorem. □

Since openness is a local condition and since the translation/rescaling of an open set is still open, to prove Brouwer's Invariance of Domain theorem, it is enough to prove the following local version:

Theorem 2.3 (Brouwer's Invariance of Domain, local version)**.** *Let $f : \overline{B^n} \to \mathbb{R}^n$ be continuous and injective. Then $f(0)$ lies in the interior of $f(\overline{B^n})$.*

A key ingredient in the proof of this "local Brouwer's Invariance of Domain theorem" is the following simple consequence of Brouwer's Fixed Point Theorem:

Lemma 2.4. *Let $f : \overline{B^n} \to \mathbb{R}^n$ be any continuous map. Suppose $h : f(\overline{B^n}) \to \mathbb{R}^n$ is continuous, and is "closed to f^{-1}" in the following sense:*

$$|h(f(x)) - x| \leqslant 1, \quad \forall x \in \overline{B^n}.$$

Then there exists $x_0 \in \overline{B^n}$ s.t. $h(f(x_0)) = 0$.

Proof. Consider the continuous map

$$\widetilde{h} : \overline{B^n} \to \mathbb{R}^n, x \mapsto x - h(f(x)).$$

By assumption, $\widetilde{h}(\overline{B^n}) \subset \overline{B^n}$. So according to Brouwer's Fixed Point Theorem, there exists $x_0 \in \overline{B^n}$ s.t. $\widetilde{h}(x_0) = x_0$, i.e. $h(f(x_0)) = 0$. □

Now we prove the "Local Brouwer's Invariance of Domain Theorem".

Proof of Local Brouwer's Invariance of Domain (following Kulpa and Tao).

> Idea: Suppose $f(0)$ is not an interior point, i.e. there exists points outside $f(\overline{B^n})$ which is arbitrarily close to $f(0)$. We will construct a map $h : f(\overline{B^n}) \to \mathbb{R}^n$ that is "closed to f^{-1}" but $h \circ f$ has "no zero", which will contradict with Lemma 2.4. To do so, we first construct a nice map that approximate f^{-1}, then perturb it to avoid zero in $f(\overline{B^n})$.

Suppose $f(0)$ is NOT an interior point of $f(\overline{B^n})$. Then for any $\varepsilon > 0$ (which we will choose later), there exists $c \in \mathbb{R}^n \setminus f(\overline{B^n})$ s.t. $|c - f(0)| < \varepsilon$. Denote

$$\Sigma_1 = \{y \in f(\overline{B^n}) : |y - c| \geqslant \varepsilon\}, \quad \Sigma_2 = \{y \in \mathbb{R}^n : |y - c| = \varepsilon\}$$

Then $\Sigma := \Sigma_1 \cup \Sigma_2$ is compact, and $f(0) \notin \Sigma$.

By assumption, $f : \overline{B^n} \to f(\overline{B^n})$ is an invertible continuous map from a compact set to a Hausdorff set. So f is a homeomorphism, i.e. it has a continuous inverse $f^{-1} : f(\overline{B^n}) \to \overline{B^n}$. Since $f(\overline{B^n})$ is compact, and thus closed, by Tietze extension theorem, there exists a continuous map $g : \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$g = f^{-1} \quad \text{on} \quad f(\overline{B^n}).$$

By construction above,

$$f(0) \notin \Sigma_1 \quad \text{and} \quad \Sigma_1 \subset f(\overline{B^n}).$$

So $g \neq 0$ on Σ_1. By continuity of g and compactness of Σ_1, there exists $0 < \delta < \frac{1}{2}$ s.t.

$$|g(y)| \geqslant \delta, \quad \forall y \in \Sigma_1.$$

By Stone-Weierstrass theorem, there exists a "polynomial map" $p : \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$|p(y) - g(y)| < \frac{\delta}{2}, \quad \forall y \in \Sigma.$$

In particular $p(y) \neq 0$ for all $y \in \Sigma_1$. However, p could be zero on Σ_2. To solve this issue, we would like to "perturb p" a little bit.

Fact 1. There exists $a_0 \in B(0, \frac{\delta}{2})$ s.t. $a_0 \notin p(\Sigma_2)$.

Assuming this. We define $\widetilde{p} = p - a_0$, then $\widetilde{p} \neq 0$ on Σ since

- $|\widetilde{p}(y) - g(y)| < \delta \Rightarrow \widetilde{p}(y) \neq 0, \forall y \in \Sigma_1$.
- $\widetilde{p}(y) \neq 0$ for $y \in \Sigma_2$, by construction.

Since there are still some points in $f(\overline{B^n})$ which are not in Σ, we define

$$\Phi : f(\overline{B^n}) \to \Sigma, \quad y \mapsto \begin{cases} y, & |y-c| \geqslant \varepsilon, \\ c + \varepsilon \frac{y-c}{|y-c|}, & |y-c| \leqslant \varepsilon. \end{cases}$$

Since $c \notin f(\overline{B^n})$, we see Φ is well-defined and continuous. So if we let

$$h : f(\overline{B^n}) \to \mathbb{R}^n, \quad y \mapsto \tilde{p}(\Phi(y)),$$

then h is continuous and $h(y) \neq 0, \forall y \in f(\overline{B^n})$.

Fact 2. We have $|h(f(x)) - x| \leqslant 1, \forall x \in \overline{B^n}$.

Thus by Lemma 2.4, there exists $x_0 \in \overline{B^n}$ s.t. $h(f(x_0)) = 0$, a contradiction! □

It remains to prove Fact 1 and Fact 2.

Proof of Fact 1.

[*Measure-theoretical proof.*] Since p is a polynomial, it is C^1. So $\exists A > 0$ s.t.

$$|p(y_1) - p(y_2)| \leqslant A|y_1 - y_2|$$

holds for any $y_1, y_2 \in B(c, 2\varepsilon)$. As a consequence, for any box

$$B := [a_1, b_1] \times \cdots \times [a_n, b_n] \subset B(c, 2\varepsilon),$$

we have

$$\mathrm{Vol}(p(B)) \leqslant A^n \mathrm{Vol}(B).$$

On the other hand, we can cover $\Sigma_2 = \partial B(c, \varepsilon)$ by finitely many boxes $B_1, \cdots, B_m$ s.t.

$$\mathrm{Vol}(B_1) + \cdots + \mathrm{Vol}(B_m) \leqslant \frac{1}{A^n} \cdot \frac{1}{2} \mathrm{Vol}(B(0, \frac{\delta}{2})).$$

As a consequence, we get

$$\mathrm{Vol}(p(\Sigma_2)) \leqslant \frac{1}{2} \mathrm{Vol}(B(0, \frac{\delta}{2})).$$

So there exists $a_0 \in B(0, \frac{\delta}{2})$ s.t. $a_0 \notin p(\Sigma_2)$. □

Proof of Fact 2.

We fix ε small enough s.t.

$$|y - f(0)| < 2\varepsilon \Longrightarrow |g(y)| = |g(y) - g(f(0))| < \frac{1}{4}.$$

Note that by definition,

$$|\tilde{p}(y) - g(y)| < \delta, \quad \forall y \in \Sigma.$$

If $y = f(x) \in \Sigma_1$, then

$$|h(f(x)) - x| = |\tilde{p}(\Phi(f(x))) - g(f(x))| = |\tilde{p}(f(x)) - g(f(x))| < \delta < \frac{1}{2}.$$

If $y = f(x) \notin \Sigma_1$, i.e. $|y - c| < \varepsilon$, we have:

- $|g(y)| < \frac{1}{4}$ since $|y - f(0)| \leqslant |y - c| + |c - f(0)| < 2\varepsilon$,
- $|g(\Phi(y))| < \frac{1}{4}$ since $|\Phi(y) - f(0)| \leqslant |\Phi(y) - c| + |c - f(0)| < 2\varepsilon$,

thus

$$\begin{aligned} |h(f(x)) - x| &= |h(y) - g(y)| \\ &\leqslant |\widetilde{p}(\Phi(y)) - g(\Phi(y))| + |g(\Phi(y)) - g(y)| \\ &< \delta + \frac{1}{4} + \frac{1}{4} < 1. \end{aligned}$$

This is what we need. □

In Lecture 12 we defined "the best" class of topological spaces, manifolds, without explicitly mention the dimension. Now we can add dimension into the definition:

Definition 2.5. Let X be a topological space. We say X is a *topological manifold of dimension n* if

(1) (T_2) X is Hausdorff;
(2) (A_2) X is 2^{nd} countable;
(3) (locally Euclidean) For any $x \in X$, there exists an open neighbourhood $U \subset X$ of x and an open set $V \subset \mathbb{R}^n$ such that $U \simeq V$.

According to the invariance of domain theorem, the dimension of a topological manifold is well-defined.

Remark. By definition, any topological manifold is locally compact, paracompact, locally path connected, and semi-locally simply connected. In PSet 5-1-3 you also proved that they are metrizable and can be embedded into higher dimensional Euclidean space.

Example. Here are some simple examples:

- S^{n-1} is an $(n-1)$-dim manifold.
- B^n is an n-dim manifold.
- $\mathbb{R}^n$ is an $n-$dim manifold.
- $S^n \times S^m$ is an $(m+n)$-dim manifold.
- $S^2 \vee S^1$ is not a topological manifold.
- $\mathbb{RP}^n$ is a topological manifold of dimension n.[PSet 5-1-3(d)]
- The connected sum of two n-dim topological manifolds is again an n-dim topological manifold.

Any 1-dimensional topological manifold is called a curve, and any 2-dimensional topological manifold is called a surface. They will be our major objects for the rest of this semester.