LECTURE 2: SMOOTH MANIFOLDS

1. Smooth Manifolds: The definition

¶ Smooth functions and smooth maps.

Let U be an open set in $\mathbb{R}^n$, and $f : U \to \mathbb{R}$ a continuous function. Recall that f is said to be a C^k*-function*, if all its partial derivatives of order at most k,

$$\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x^\alpha} := \frac{\partial^{|\alpha|} f}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$$

exist and are continuous on U. We say that f is a C^∞ function, or a *smooth function*, if it is of class C^k for all positive integers k. A function f is an *analytic function* (or a C^ω function) if it is smooth and agrees with its Taylor series in a neighborhood of every point. Note that not all smooth functions are analytic.

Now let U be an open set in $\mathbb{R}^n$ and V an open set in $\mathbb{R}^m$. Let

$$f = (f_1, \cdots, f_m) : U \to V$$

be a continuous map. We say f is C^∞ (or C^k, or C^ω) if each component f_i, $1 \leq i \leq m$, is a C^∞ (or C^k, or C^ω) function. [1]

Definition 1.1. A smooth map $f : U \to V$ is a *diffeomorphism* if f is one-to-one and onto, and $f^{-1} : V \to U$ is also smooth.

Obviously

- If $f : U \to V$ is a diffeomorphism, so is f^{-1}.
- If $f : U \to V$ and $g : V \to W$ are diffeomorphisms, so is $g \circ f : U \to W$.

¶ Definition of smooth manifolds.

We would like to define smooth structures on topological manifolds so that one can do calculus on it. In particular, we should be able to talk about smoothness of continuous functions on a given smooth manifold M. Since near each point in M, one has a chart $\{\varphi, U, V\}$ which identify the open set U in M with the open set V in $\mathbb{R}^n$, it is natural identify any function f on U with the function $f \circ \varphi^{-1}$ on V, and use the smoothness of $f \circ \varphi^{-1}$ to define the smoothness of f itself. This idea is of course correct. The only issue is that a point on M could sit in many different open charts, and the smoothness of a function at this point should be independent of the choice

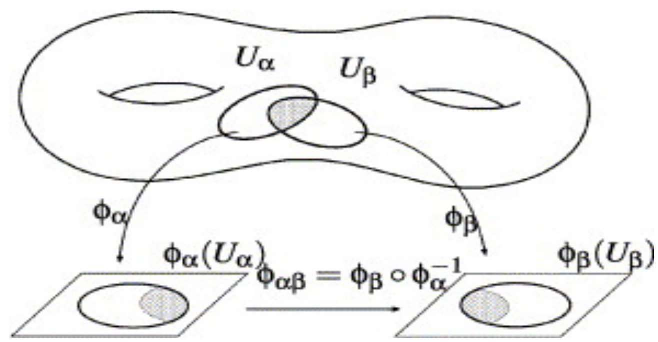
[1]In this course we will mainly consider C^∞ functions/maps. However, most definitions/theorems can be easily extended to the C^k setting. On the other hand, C^ω theory will be quite different.

of charts. In other words, if both φ and ψ are chart maps near a point, we want the maps $f \circ \varphi^{-1}$ and $f \circ \psi^{-1}$ to be simultaneously smooth or non-smooth. This amounts to require $\varphi \circ \psi^{-1}$ to be smooth. (Note: even though $f \circ \varphi^{-1}$ and $f \circ \psi^{-1}$ are both smooth, we still want the map $\varphi \circ \psi^{-1}$ to be smooth so that the differentials of $f \circ \varphi^{-1}$ and $f \circ \psi^{-1}$ are nicely related by the chain rule.) With this requirement at hand, we define

Definition 1.2. Let M be a topological manifold of dimension n. We say two charts $\{\varphi_\alpha, U_\alpha, V_\alpha\}$ and $\{\varphi_\beta, U_\beta, V_\beta\}$ of M are *compatible* if the *transition map*

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism. [Note that both $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$, so the smoothness of $\varphi_{\alpha\beta}$ is well understood.]



Definition 1.3. (1) An *atlas* $\mathcal{A}$ on M is a collection of charts $\{\varphi_\alpha, U_\alpha, V_\alpha\}$ with $\bigcup_\alpha U_\alpha = M$, such that all charts in $\mathcal{A}$ are compatible to each other.
(2) Two atlas on M are said to be *equivalent* if their union is still an atlas on M.

Example. We can define three atlas on $\mathbb{R}$ by $\mathcal{A}_i = \{\varphi_i, \mathbb{R}, \mathbb{R}\}$ $(1 \le i \le 3)$, where $\varphi_1(x) = x$, $\varphi_2(x) = 2x$ and $\varphi_2(x) = x^3$. Then $\mathcal{A}_1, \mathcal{A}_2$ are equivalent, but $\mathcal{A}_1, \mathcal{A}_3$ are non-equivalent since

$$\varphi_{31}(x) = \varphi_1 \circ \varphi_3^{-1}(x) = x^{1/3}$$

is not smooth on $\mathbb{R}$.

Definition 1.4. An n-dimensional *smooth manifold* is an n-dimensional topological manifold M equipped with an equivalence class of atlas. This equivalence class is called its *smooth structure.*

So a smooth manifold is a pair $(M, \mathcal{A})$. In the future we will always omit $\mathcal{A}$ if there is no confusion of the smooth structure.

Remark. Similarly one can define C^k manifolds, real analytic $(=C^\omega)$ manifolds and complex manifolds. For example, a complex manifold is a Hausdorff and second countable topological space that locally looks like $\mathbb{C}^n$, so that the transition maps are all holomorphic maps.

Remark. One can also define and study infinitely dimensional manifolds. There are many different theories on infinitely dimensional manifolds, depending on whether the manifold is modelled locally on a Banach space, a Hilbert space, a Fréchet space etc, and one can define smooth structures on such manifolds.

Remark. Some deep results from differential topology:

- There exist topological manifolds that do not admit smooth structure. The first example is a compact 10-dimensional manifold found by M. Kervaire.
- If a topological manifold admits a C^1 structure, it also admits a C^∞ structure.
- Any manifold M admits a finite atlas consisting of $\dim M + 1$ charts (not necessarily connected).

2. Examples of Smooth Manifolds

¶ First examples.

Note that by definition, we immediately have

Proposition 2.1. *If a topological manifold M can be covered by a single chart, then such a chart automatically determines a smooth structure on M.*

As a consequence,

- $\mathbb{R}^n$ and any open subset of $\mathbb{R}^n$ is a smooth manifold.
- The general linear group $GL(n, \mathbb{R})$ that we studied last time is a smooth manifold.

Example. **(Graphs).** For any open set $U \subset \mathbb{R}^m$ and any continuous map $f : U \to \mathbb{R}^n$, the *graph* of f is the subset in $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ defined by

$$\Gamma(f) = \{(x, y) \mid x \in U, y = f(x)\} \subset \mathbb{R}^{m+n}.$$

With the subspace topology inherited from $\mathbb{R}^{m+n}$, $\Gamma(f)$ is Hausdorff and second-countable. It is locally Euclidean since it has a global chart $\{\varphi, \Gamma(f), U\}$, where

$$\varphi : \Gamma(f) \to U, \quad \varphi(x, y) = x$$

is the *projection onto the first factor* map. [To see why φ is a homeomorphism: obviously φ is continuous, invertible, and its inverse

$$\varphi^{-1} : U \to \Gamma(f), \quad \varphi^{-1}(x) = (x, f(x))$$

is continuous.] So $\Gamma(f)$ is a topological manifold of dimension m. Since it can be covered by one chart, we conclude

> The graph $\Gamma(f)$ of any continuous function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ admits an intrinsic structure of a smooth manifold. [However, it is possible that $\Gamma(f)$ is not a *smooth submanifold* of $\mathbb{R}^{n+1}$.]

¶ The spheres as smooth manifolds.

Example. **(Spheres).** For each $n \geq 0$, the unit n-sphere

$$S^n = \{(x^1, \cdots, x^n, x^{n+1}) \mid (x^1)^2 + \cdots + (x^n)^2 + (x^{n+1})^2 = 1\} \subset \mathbb{R}^{n+1}$$

with the subspace topology is Hausdorff and second-countable. To show that it is locally Euclidean, we can cover S^n by two open subsets

$$U_+ = S^n \setminus \{(0, \cdots, 0, -1)\}, \qquad U_- = S^n \setminus \{(0, \cdots, 0, 1)\}$$

and define two charts $\{\varphi_+, U_+, \mathbb{R}^n\}$ and $\{\varphi_-, U_-, \mathbb{R}^n\}$ by the *stereographic projections*

$$\varphi_\pm(x^1, \cdots, x^n, x^{n+1}) = \frac{1}{1 \pm x^{n+1}}(x^1, \cdots, x^n).$$

It is easy to check that $\varphi_\pm$ are continuous, invertible, and the inverse

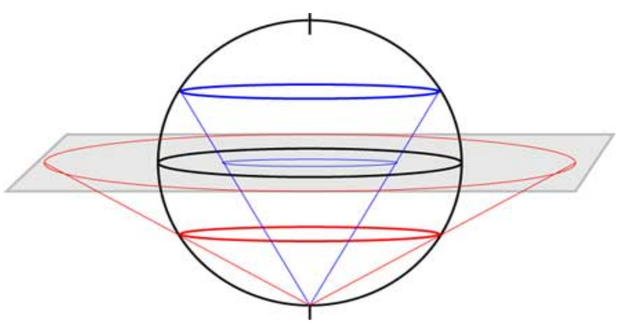
$$\varphi_\pm^{-1}(y^1, \cdots, y^n) = \frac{1}{1 + (y^1)^2 + \cdots + (y^n)^2}\left(2y^1, \cdots, 2y^n, \pm(1 - (y^1)^2 - \cdots - (y^n)^2)\right)$$

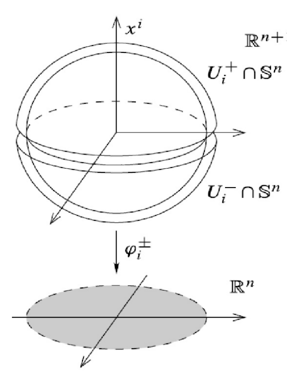
is also continuous.

It follows that on $\varphi_-(U_+ \cap U_-) = \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned}
\varphi_{-+}(y^1, \cdots, y^n) &= \varphi_+ \circ \varphi_-^{-1}(y^1, \cdots, y^n) \\
&= \varphi_+\left(\frac{1}{1 + |y|^2}\left(2y^1, \cdots, 2y^n, -1 + |y|^2\right)\right) \\
&= \frac{1}{|y|^2}(y^1, \cdots, y^n),
\end{aligned}$$

which is a diffeomorphism from $\mathbb{R}^n \setminus \{0\}$ to itself. So these two charts are compatible.





Remark. We can also cover S^n by $2n + 2$ charts using hemispheres. More precisely, for any $1 \leq i \leq n + 1$, we let

$$U_i^+ = \{(x^1, \cdots, x^{n+1}) \in S^n : x^i > 0\}$$

be the "upper hemisphere" in the i^{th} direction and define $\varphi_i^+ : U_i^+ \to B^n(1)$ be the projection map

$$\varphi_i^+(x^1, \cdots, x^{n+1}) = (x^1, \cdots, x^{i-1}, x^{i+1}, x^{n+1}),$$

where $B^n(1)$ is the unit ball in $\mathbb{R}^n$. Then one can check that $(\varphi_i^+, U_i^+, B^n(1))$ are charts. Similarly one can construct charts $(\varphi_i^-, U_i^-, B^n(1))$ on each "lower hemispheres". [Check: charts of S^n defined via hemispheres are compatible with these two charts.]

¶ The real projective spaces as smooth manifolds.

Example. **(The Real Projective Spaces).**

The n dimensional *real projective space* $\mathbb{RP}^n$ is by definition the set of lines passing the origin in $\mathbb{R}^{n+1}$. To give $\mathbb{RP}^n$ a topology, we will regard it as the quotient space

$$\mathbb{RP}^n = \mathbb{R}^{n+1} - \{(0, \cdots, 0)\}/\sim,$$

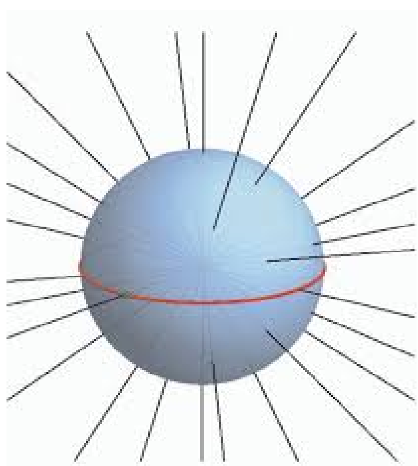
where the equivalent relation $\sim$ is given by

$$(x^1, \cdots, x^{n+1}) \sim (tx^1, \cdots, tx^{n+1}), \qquad \forall t \neq 0.$$

One can also regard $\mathbb{RP}^n$ as the quotient of S^n by gluing the antipodal points

$$\mathbb{RP}^n = S^n/\sim.$$

Form these descriptions it is easy to see that $\mathbb{RP}^n$ is Hausdorff and second-countable, and in fact is compact.[Prove it.]



Usually people will denote the element (=the equivalence class or the "line") in $\mathbb{RP}^n$ containing the point $(x^1, \cdots, x^{n+1})$ by $[x^1 : \cdots : x^{n+1}]$.

Now we construct local charts on $\mathbb{RP}^n$. Consider the open sets

$$U_i = \{[x^1 : \cdots : x^{n+1}] \mid x^i \neq 0\}, \qquad 1 \leq i \leq n+1.$$

For each i, define $\varphi_i : U_i \to \mathbb{R}^n$ to be

$$\varphi_i([x^1 : \cdots : x^{n+1}]) = \left(\frac{x^1}{x^i}, \cdots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \cdots, \frac{x^{n+1}}{x^i}\right).$$

It is not hard to check that each φ_i is well-defined, is continuous, and the inverse

$$\varphi_i^{-1}(y^1, \cdots, y^n) = [y^1 : \cdots : y^{i-1} : 1 : y^i : \cdots : y^n].$$

is continuous. So each $(\varphi_i, U_i, \mathbb{R}^n)$ is a chart and $\mathbb{RP}^n$ is a topological manifold.

Finally we will show that $\mathbb{RP}^n$ is in fact a smooth manifold. Without loss of generality, let's verify that $\varphi_{1,n+1}$ is a diffeomorphism between

$$\varphi_1(U_1 \cap U_{n+1}) = \{(y^1, \cdots, y^n) \mid y^n \neq 0\} =: V_n$$

and

$$\varphi_{n+1}(U_1 \cap U_{n+1}) = \{(y^1, \cdots, y^n) \mid y^1 \neq 0\} =: V_1.$$

In fact, by definition

$$\begin{aligned}\varphi_{1,n+1}(y^1, \cdots, y^n) &= \varphi_{n+1} \circ \varphi_1^{-1}(y^1, \cdots, y^n) \\ &= \varphi_{n+1}([1 : y^1 : \cdots : y^n]) \\ &= \left(\frac{1}{y^n}, \frac{y^1}{y^n}, \cdots, \frac{y^{n-1}}{y^n}\right)\end{aligned}$$

which is obviously a diffeomorphism from V_n to V_1. Similarly one can show that all other transition maps φ_{ij} are diffeomorphisms.

Remark. By a similar way one can define the n dimensional *complex projective space* $\mathbb{CP}^n$ as the space of "complex lines" in $\mathbb{C}^n$ and verify that it is a smooth manifold. More generally, one can show that Grassmannian [2] $Gr(k, V)$, the space of all k-dimensional linear subspaces of an n-dimensional (real or complex) vector space V, is a smooth manifold. For details, c.f. John Lee, page 22-24.

Example. **(The set of all straight lines in $\mathbb{R}^2$).** The set of all straight lines in $\mathbb{R}^2$ is a manifold. To see this, we just notice that any straight line is of the form

$$ax + by + c = 0$$

for some $a, b, c \in \mathbb{R}$, with two triples (a, b, c) and (a', b', c') defines the same line if and only if $[a : b : c] = [a' : b' : c']$, that is, if and only if they give the same point in $\mathbb{RP}^2$. Also notice that $[0 : 0 : 1]$ will not give us a line, while each other element in $\mathbb{RP}^2$ gives us a line in $\mathbb{R}^2$. Thus we get a well-defined bijective map [the image is a Möbius band!]

$$\phi : \{\text{the set of lines in } \mathbb{R}^2\} \to \mathbb{RP}^2 \setminus \{[0 : 0 : 1]\},$$
$$ax + by + c = 0 \mapsto [a : b : c].$$

Now we just "move" all structures on $\mathbb{RP}^2$ to the set of all straight lines in $\mathbb{R}^2$ and thus define a smooth manifold structure.[Try to construct manifold structure by local parametrizations directly.]

[2] Named after H. Grassmann, the founder of linear algebra!