LECTURE 3: SMOOTH FUNCTIONS; PARTITION OF UNITY

1. Smooth Functions

¶ Smooth functions on manifolds.

Definition 1.1. Let $(M, \mathcal{A})$ be a smooth manifold, and $f : M \to \mathbb{R}$ a function.

(1) We say f is *smooth at* $p \in M$ if there exists a chart $(\varphi_\alpha, U_\alpha, V_\alpha) \in \mathcal{A}$ with $p \in U_\alpha$, such that the function $f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{R}$ is smooth at $\varphi_\alpha(p)$.

(2) We say f is a *smooth function* on M if it is smooth at every $x \in M$.

Remark. Suppose $f \circ \varphi_\alpha^{-1}$ is smooth at $\varphi(p)$. Let $(\varphi_\beta, U_\beta, V_\beta)$ be another chart in $\mathcal{A}$ with $p \in U_\beta$. Then by the compatibility of charts, the function

$$f \circ \varphi_\beta^{-1} = (f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi_\beta^{-1})$$

must be smooth at $\varphi_\beta(p)$. So the smoothness of a function is independent of the choice of charts in the given atlas.

Remark. According to the chain rule, it is easy to see that if $f : M \to \mathbb{R}$ is smooth at $p \in M$, and $h : \mathbb{R} \to \mathbb{R}$ is smooth at $f(p)$, then $h \circ f$ is smooth at p.

Example. Each coordinate function $f_i(x^1, \cdots, x^{n+1}) = x^i$ is smooth on S^n since

$$f_i \circ \varphi_\pm^{-1}(y^1, \cdots, y^n) = \begin{cases} \frac{2y^i}{1+|y|^2}, & 1 \le i \le n \\ \pm\frac{1-|y|^2}{1+|y|^2}, & i = n+1 \end{cases}$$

are smooth functions on $\mathbb{R}^n$. Similarly the latitude is a smooth function on S^2, since it can be written as the composition of the "height function" x^3 with a smooth function. However, the longitude is not even a well-defined real-valued function on S^2.

We will denote the set of all smooth functions on M by $C^\infty(M)$. Note that this is a (commutative) algebra, i.e. it is a vector space equipped with a (commutative) bilinear "multiplication operation": If f, g are smooth, so are $af + bg$ and fg; moreover, the multiplication is commutative, associative and satisfies the usual distributive laws.

Now suppose $f \in C^\infty(M)$. As usual, the *support* of f is by definition the set

$$\mathrm{supp}(f) = \overline{\{p \in M \mid f(p) \neq 0\}}.$$

We say that f is *compactly supported*, denoted by $f \in C_0^\infty(M)$, if the support of f is a compact subset in M. Obviously

- if $f, g \in C_0^\infty(M)$, then $af + bg \in C_0^\infty(M)$.
- if $f \in C_0^\infty(M)$ and $g \in C^\infty(M)$, then $fg \in C_0^\infty(M)$.

So $C_0^\infty(M)$ is an ideal of of the algebra $C^\infty(M)$. Note that if M is compact, then any smooth function is compactly supported.

¶ Bump functions.

A *bump function* (sometimes also called a *test function*) is a compactly supported smooth function, which is usually supposed to be non-negative, no more than 1, and equals to 1 on a given compact set (or has total integral 1 on a given set).

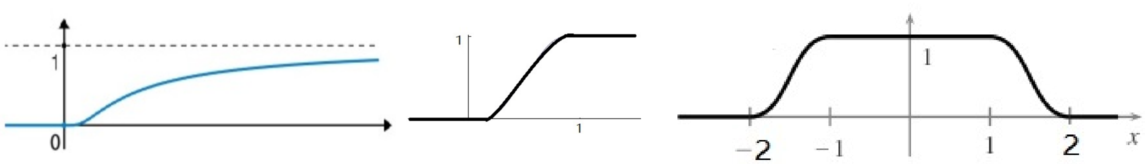
Example (A bump function on $\mathbb{R}^n$). In what follows we first define two auxiliary functions f_1 and f_2 on $\mathbb{R}$. Then we define a bump function f_3 on $\mathbb{R}^n$. We list the definition of f_k in the left, and list the properties of f_k in the right. The smoothness and properties of f_k follows from those of f_{k-1} (so you should check f_1 is a smooth function):

$$f_1(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \le 0 \end{cases} \qquad \Longrightarrow f_1(x) = \begin{cases} \in (0,1), & x > 0, \\ 0, & x \le 0, \end{cases}$$

$$f_2(x) = \frac{f_1(x)}{f_1(x) + f_1(1-x)} \qquad \Longrightarrow f_2(x) = \begin{cases} 0, & x \le 0, \\ \in (0,1), & 0 < x < 1, \\ 1, & x \ge 1 \end{cases}$$

$$f_3(x) = f_2(2 - |x|) \qquad \Longrightarrow f_3(x) = \begin{cases} 0, & |x| \ge 2, \\ \in (0,1), & 1 < |x| < 2, \\ 1, & |x| \le 1. \end{cases}$$

The graphs of f_1, f_2 and f_3 (with $n = 1$) are shown below:



With the help of this Euclidean bump function, we can construct bump functions on any smooth manifold with prescribed support and prescribed "equal to one region":

Theorem 1.2. *Let M be a smooth manifold, $A \subset M$ is a compact subset, and $U \subset M$ an open subset that contains A. Then there is a bump function $\varphi \in C_0^\infty(M)$ so that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on A and* $\mathrm{supp}(\varphi) \subset U$.

Proof. [The idea of the proof: Cover the compact set A by finitely many small pieces, where each piece is contained in one (carefully chosen) chart, so that one can copy the "Euclidean bump function" that we constructed above to such pieces.]

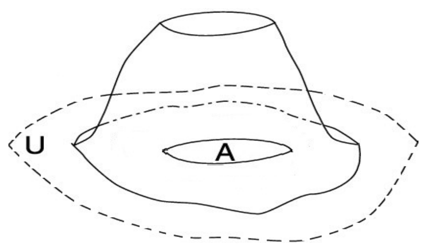
For each $q \in A$, there is a chart (φ_q, U_q, V_q) near q so that $U_q \subset U$ and V_q contains the open ball $B_3(0)$ of radius 3 centered at 0 in $\mathbb{R}^n$. (Question: Why one can find such a chart which is compatible with given charts?) Let $\widetilde{U}_q = \varphi_q^{-1}(B_1(0))$, and let

$$f_q(p) = \begin{cases} f_3(\varphi_q(p)), & p \in U_q, \\ 0, & p \notin U_q. \end{cases}$$

Then $f_q \in C_0^\infty(M)$, $\mathrm{supp}(f_q) \subset U_p$ and $f_q \equiv 1$ on $\widetilde{U}_q$. (Question: which assumption on manifold do we need here?)

Now the family of open sets $\{\widetilde{U}_q\}_{q\in A}$ is an open cover of A. Since A is compact, there is a finite sub-cover $\{\widetilde{U}_{q_1},\cdots,\widetilde{U}_{q_N}\}$. Let $\psi=\sum_{i=1}^N f_{q_i}$. Then ψ is a compactly supported smooth function on M so that $\psi\geq 1$ on A and $\mathrm{supp}(\psi)\subset U$. It follows that the function $\varphi(p)=f_2(\psi(p))$ satisfies all the conditions we want. □

Here is what such a bump function will look like:



As a simple consequence, we see that the vector space $C_0^\infty(M)$ (and thus $C^\infty(M)$) is infinitely dimensional (assuming $\dim M>0$).

2. Partition of unity

¶ Partition of unity.

So for any compact subset $K\subset M$, one can always cover it by finitely many nice neighborhoods on which we can construct nice "local" functions. By adding these (finitely many) local functions, we can get nice global functions on M that behaves nicely on K. It turns out that the same idea applies to the whole manifold M: we can generate an infinite collection of smooth functions on M, and add them to get a global smooth function, provided that near each point, there are only finitely many nonzero functions in our collection. More importantly, we can use such a collection of functions to "glue" geometric/analytic objects that can be defined locally using charts.

Definition 2.1. Let M be a smooth manifold, and $\{U_\alpha\}$ an open cover of M. A (smooth) *partition of unity* (P.O.U. in brief)[1] *subordinate to the cover* $\{U_\alpha\}$ is a collection of smooth functions $\{\rho_\alpha\}$ defined on the whole manifold M so that

(1) $0\leq\rho_\alpha\leq 1$ for all α.
(2) $\mathrm{supp}(\rho_\alpha)\subset U_\alpha$ for all α.
(3) each $p\in M$ has a neighborhood which intersects only finitely many $\mathrm{supp}(\rho_\alpha)$'s.
(4) $\sum_\alpha\rho_\alpha(p)=1$ for all $p\in M$.

Remark. Two consequences of the local finiteness condition (3): Let's denote by W_p a neighborhood of p which intersect only finitely many $\mathrm{supp}(\rho_\alpha)$'s. Then

[1]In this course, when we talk about P.O.U., we always mean smooth partition of unity. See my general topology course notes for a theory of continuous partition of unity for paracompact spaces. Recall that a topological space X is *paracompact* if every open covering admits a locally finite open refinement. It is not hard to show each topological manifold is paracompact.

- Since $\{W_p\}_{p\in M}$ is an open cover of M, and since M is second countable, one can find countably many W_{p_i}'s which also cover M. Since each W_{p_i} intersect only finitely many $\mathrm{supp}(\rho_\alpha)$'s, we conclude that there are only countable many ρ_α's whose support are non-empty. So even if we may start with uncountably many open sets, the P.O.U. automatically "delete" most of them so that only countably many of them are left (which still form an open cover of M).
- For each p, on the open set W_p, a sum like (4) [which looks like an uncountable sum, or maybe a countable infinite sum in view of the previous paragraph] is in fact a finite sum. This fact is CRUCIAL in applications.

The main result in this section is to prove

Theorem 2.2 (The existence of P.O.U.). *Let M be a smooth manifold, and $\{U_\alpha\}$ an open cover of M. Then there exists a P.O.U. subordinate to $\{U_\alpha\}$.*

Locally each manifold looks like $\mathbb{R}^n$, so that one have rich mathematics on it. P.O.U. is the tool that can "glue" local smooth objects into a global smooth object. We will see many such examples in the future. For example, we will

- approximate continuous functions/maps via smooth functions/maps.
- define integrals of differential forms in local charts, and use P.O.U. to define the integral of a differential form on the whole manifold.
- (in future course) construct Riemannian metric, linear connection etc.

As an immediate application of P.O.U., we generalize Theorem 1.2 to closed subsets:

Corollary 2.3. *Let M be a smooth manifold, $A \subset M$ is a closed subset, and $U \subset M$ an open subset that contains A. Then there is a "bump" function $\varphi \in C^\infty(M)$ so that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on A and $\mathrm{supp}(\varphi) \subset U$.*

Proof. Obviously $\{U, M \setminus A\}$ is an open covering of M. Let $\{\rho_1, \rho_2\}$ be a P.O.U. subordinate to this open covering. Then the function $\varphi = \rho_1$ is what we need: ρ_1 is smooth, $0 \le \rho_1 \le 1$, $\mathrm{supp}(\rho_1) \subset U$, and finally $\rho_1 = 1$ on A since $\rho_2 = 0$ on A. □

Note that this implies a smooth version of Urysohn's lemma [See today's PSet].

¶ Existence of P.O.U.: The proof.

The proof relies on the following technical lemma from general topology.

Lemma 2.4. *For any open covering $\mathcal{U} = \{U_\alpha\}$ of a topological manifold M, one can find two countable family of open covers $\mathcal{V} = \{V_j\}$ and $\mathcal{W} = \{W_j\}$ of M so that*

- *For each j, $\overline{V}_j$ is compact and $\overline{V}_j \subset W_j$.*
- *$\mathcal{W}$ is a* refinement *of $\mathcal{U}$: For each j, there is an $\alpha = \alpha(j)$ so that $W_j \subset U_\alpha$.*
- *$\mathcal{W}$ is* locally finite*: Any $p \in M$ has a neighborhood W such that $W \cap W_j \neq \varnothing$ for only finitely many W_j's.*

We will first prove Theorem 2.2 and postpone the proof Lemma 2.4 as an appendix.

Proof of Theorem 2.2. [Please compare the first paragraph of this proof with the proof of Theorem 1.2.] Since $\overline{V}_j$ is compact and $\overline{V}_j \subset W_j$, according to Theorem 1.2 we can find nonnegative functions $\varphi_j \in C_0^\infty(M)$ so that

$$0 \le \varphi_j \le 1, \qquad \varphi_j \equiv 1 \text{ on } \overline{V}_j, \qquad \text{supp}(\varphi_j) \subset W_j.$$

Since $\mathcal{W}$ is a locally finite covering, the function

$$\varphi = \sum_j \varphi_j$$

is a well-defined smooth function on M. Since each φ_j is nonnegative, and $\mathcal{V}$ is a covering of M, φ is strictly positive on M. It follows that the functions

$$\psi_j = \frac{\varphi_j}{\varphi}$$

are smooth and satisfy $0 \le \psi_j \le 1$ and $\sum_j \psi_j = 1$.

Next let's re-index the family $\{\psi_j\}$ to get the demanded P.O.U. subordinate to $\{U_\alpha\}$. For each j, we fix an index $\alpha(j)$ so that $W_j \subset U_{\alpha(j)}$, and define

$$\rho_\alpha = \sum_{\alpha(j)=\alpha} \psi_j.$$

Note that the right hand side is a finite sum near each point, so it does define a smooth function. By local finiteness of $\mathcal{W}$,

$$\text{supp}\rho_\alpha = \overline{\bigcup_{\alpha(j)=\alpha} \text{supp}\psi_j} = \bigcup_{\alpha(j)=\alpha} \overline{\text{supp}\psi_j} = \bigcup_{\alpha(j)=\alpha} \text{supp}\psi_j \subset U_\alpha.$$

Clearly the family $\{\rho_\alpha\}$ is a P.O.U. subordinate to $\{U_\alpha\}$. □

¶ Appendix: The proof of Lemma 2.4.

It remains to prove Lemma 2.4. In particular, we want to prove the existence of locally finite open refinement. The proof is quite geometric. First we prove

Lemma 2.5. *For any topological manifold M, there exists a countable collection of open sets $\{X_i\}$ so that* [*Such a collection of subsets is called an* exhaustion *of M*]

(1) *For each j, the closure $\overline{X}_j$ is compact.*
(2) *For each j, $\overline{X}_j \subset X_{j+1}$.*
(3) *$M = \cup_j X_j$.*

Proof. Since M is second countable, there is a countable basis of the topology of M. Out of this countable collection of open sets, we pick those that have compact closures, and denote them by $Y_1, Y_2, \cdots$. Since M is locally Euclidean, it is easy to see that $\mathcal{Y} = \{Y_j\}$ is an open cover of M.

We let $X_1 = Y_1$. Since $\mathcal{Y}$ is an open cover of $\overline{X}_1$ which is compact, there exist finitely many open sets $Y_{i_1}, \cdots, Y_{i_k}$ so that

$$\overline{X}_1 \subset Y_{i_1} \cup \cdots \cup Y_{i_k}.$$

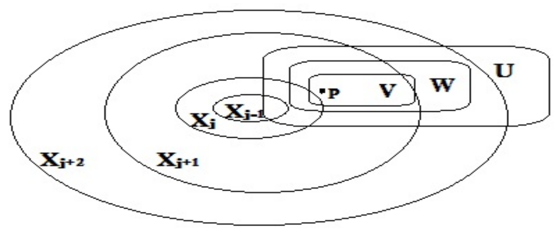
Let

$$X_2 = Y_2 \cup Y_{i_1} \cup \cdots \cup Y_{i_k}.$$

Obviously $\overline{X}_2$ is compact. By repeating this procedure we get a sequence of open sets $X_1, X_2, X_3, \cdots$ which satisfies (1) and (2). It satisfies (3) since $X_k \supset \cup_{j=1}^k Y_j$ □

Proof of Lemma 2.4. For each $p \in M$, there is an j and an $\alpha(p)$ so that $p \in \overline{X}_{j+1} \setminus X_j$ and $p \in U_{\alpha(p)}$. Since M is locally Euclidean, one can choose open neighborhoods V_p, W_p of p so that $\overline{V}_p$ is compact and

$$\overline{V}_p \subset W_p \subset U_{\alpha(p)} \cap (X_{j+2} \setminus \overline{X}_{j-1}).$$



Now for each j, since the "stripe" $\overline{X}_{j+1} \setminus X_j$ is compact, one can choose finitely many points $p_1^j, \cdots, p_{k_j}^j$ so that $V_{p_1^j}, \cdots, V_{p_{k_j}^j}$ is an open cover of $\overline{X}_{j+1} \setminus X_j$. Denote all these $V_{p_k^j}$'s by $V_1, V_2, \cdots$, and the corresponding $W_{p_k^j}$'s by $W_1, W_2, \cdots$. Then $\mathcal{V} = \{V_k\}$ and $\mathcal{W} = \{W_k\}$ are open covers of M that satisfies all the conditions in Lemma 2.4. For example, the local finiteness property of $\mathcal{W}$ follows from the fact that there are only finitely many W_k's (that correspond to j and $j-1$ above) intersect $X_{j+1} \setminus \overline{X}_{j-1}$. □

We end this section with the following question:

Where did we use the Hausdorff condition in proving P.O.U.?