LECTURE 4: SMOOTH MAPS

1. Smooth Maps

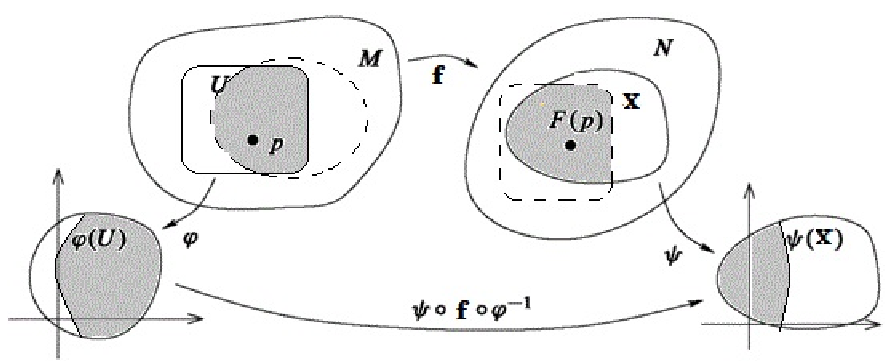
¶ Smooth maps between manifolds.

Since manifolds are locally Euclidean, we are able to transplant conceptions defined on Euclidean spaces to manifolds. Recall that a smooth function on a smooth manifold M is a function $f : M \to \mathbb{R}$ so that for any chart[1] $(\varphi_\alpha, U_\alpha, V_\alpha)$ of M, the function $f \circ \varphi_\alpha^{-1}$ is a smooth function on V_α. More generally, we can define smooth maps between smooth manifolds:

Definition 1.1. Let M, N be smooth manifolds. We say a continuous map $f : M \to N$ is *smooth* if for any chart $(\varphi_\alpha, U_\alpha, V_\alpha)$ of M and chart $(\psi_\beta, X_\beta, Y_\beta)$ of N, the map

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap f^{-1}(X_\beta)) \to \psi_\beta(X_\beta)$$

is smooth. [Again: Both $\varphi_\alpha(U_\alpha \cap f^{-1}(X_\beta))$ and $\psi_\beta(X_\beta)$ are Euclidian open sets.]



Similarly one can define C^k or C^ω maps between C^k or C^ω manifolds.

Remark. In the definition we require the map f to be continuous. This is to guarantee that the map $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is defined on a neighborhood of $\varphi_\alpha(p)$. In general the smoothness of all $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$'s does not imply the continuity of f.[2] (See Problem Set)

The following proposition (which claims that the conception of smoothness is independent of the choice of equivalent smooth structures) can be easily checked:

Proposition 1.2. *If $f : (M, \mathcal{A}) \to (N, \mathcal{B})$ is smooth, $\mathcal{A}_1$ and $\mathcal{B}_1$ are atlas on M and N that are compatible with $\mathcal{A}$ and $\mathcal{B}$ respectively, then $f : (M, \mathcal{A}_1) \to (N, \mathcal{B}_1)$ is smooth.*

[1] In this course, when we say "any chart of a smooth manifold M", we always mean "any chart in a given atlas $\mathcal{A}$ that defines the smooth structure of M".

[2] Alternatively, it is also enough to assume that each point p has a coordinate neighborhood U so that $f(U)$ is contained in a coordinate domain on N. c.f. Hirsch, Differential Topology, GTM 33.

The set of all smooth maps from M to N is denoted by $C^\infty(M,N)$. We will leave it as an exercise to prove that if $f \in C^\infty(M,N)$ and $g \in C^\infty(N,P)$, then $g \circ f \in C^\infty(M,P)$. As a consequence, any smooth map $f : M \to N$ induces a "pull-back" map

$$f^* : C^\infty(N) \to C^\infty(M), \quad g \mapsto g \circ f.$$

The pull-back will play an important role in the future.

¶ Examples of smooth maps.

Example. If we equip $\mathbb{R}$ with the standard smooth structure (i.e., the smooth structure $\{(\varphi_1(x) = x, \mathbb{R}, \mathbb{R})\}$, which we will always assume), a map $f : M \to \mathbb{R}$ is a smooth map in the definition above if and only if it is a smooth function in the sense that we learned last time. More generally, a map

$$f = (f_1, \cdots, f_k) : M \to \mathbb{R}^k$$

is a smooth map if and only if each $f_i \in C^\infty(M)$.

Example. Recall that the general linear group $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ and thus is a smooth manifold. It is then easy to check (by using the standard atlas with only one chart) that (why?)

- the determinant
$$\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}, \quad A \mapsto \det A$$
is a smooth function,
- the matrix product map
$$m : \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}), \quad (A,B) \mapsto AB$$
is a smooth map,
- the matrix inversion map
$$i : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}), \quad A \mapsto A^{-1}$$
is a smooth map.

As we will see, these facts imply that $\mathrm{GL}(n,\mathbb{R})$ is a Lie group.

Example. The inclusion map $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth, since

$$\iota \circ \varphi_\pm^{-1}(y^1, \cdots, y^n) = \frac{1}{1+|y|^2}\left(2y^1, \cdots, 2y^n, \pm(1-|y|^2)\right)$$

are smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$. Note that by definition, if g is any smooth function on $\mathbb{R}^{n+1}$, the pull-back function $\iota^* g$ is just the restriction of g to S^n:

$$\iota^* g = g|_{S^n}.$$

So the restriction of any smooth function on $\mathbb{R}^{n+1}$ to S^n is a smooth function on S^n.

Example. The projection map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is smooth, since

$$\varphi_i \circ \pi(x^1, \cdots, x^{n+1}) = \left(\frac{x^1}{x^i}, \cdots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \cdots, \frac{x^{n+1}}{x^i}\right)$$

is smooth on $\pi^{-1}(U_i) = \{(x^1, \cdots, x^{n+1}) : x^i \neq 0\}$ for each i.

¶ Diffeomorphisms.

As in the Euclidean case, we can define diffeomorphisms between smooth manifolds.

Definition 1.3. Let M, N be smooth manifolds. A map $f : M \to N$ is a *diffeomorphism* if it is smooth, bijective, and the inverse f^{-1} is smooth.

If there exists a diffeomorphism $f : M \to N$, then we say M and N are diffeomorphic. Sometimes we will denote $M \simeq N$. The following properties can be easily deduced from the corresponding properties in the Euclidean case:

- The identity map $\mathrm{Id} : M \to M$ is a diffeomorphism.
- If $f : M \to N$ and $g : N \to P$ are diffeomorphisms, so is $g \circ f$.
- If $f : M \to N$ is a diffeomorphism, so is f^{-1}. Moreover, $\dim M = \dim N$.

So "diffeomorphism" defines an equivalence relation on the set of all smooth manifolds. We will regard diffeomorphic smooth manifolds as the same.

Remark. In particular, for any smooth manifold M,

$$\mathrm{Diff}(M) = \{f : M \to M \mid f \text{ is a diffeomorphism}\}$$

is a group, which is known as the *diffeomorphism group* of M.

Example. If M is a smooth manifold, then any chart (φ, U, V) gives a diffeomorphism $\varphi : U \to V$ from $U \subset M$ (where we regard U as a smooth manifold) to $V \subset \mathbb{R}^n$. More generally, one can check: two atlas $\mathcal{A} = \{(\phi_\alpha, U_\alpha, V_\alpha)\}_\alpha$ and $\mathcal{B} = \{(\varphi_\beta, U_\beta, V_\beta)\}_\beta$ of M are equivalent if and only if the identity map $\mathrm{Id} : (M, \mathcal{A}) \to (M, \mathcal{B})$ is a diffeomorphism.

Example. We have seen that on $M = \mathbb{R}$, the two atlas $\mathcal{A} = \{(\varphi_1(x) = x, \mathbb{R}, \mathbb{R})\}$ and $\mathcal{B} = \{(\varphi_2(x) = x^3, \mathbb{R}, \mathbb{R})\}$ define non-equivalent smooth structures. However, the map

$$f : (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}), \quad f(x) = x^{1/3}$$

is a diffeomorphism. So we still think these two smooth structures are the same, although they are not equivalent.

Remark. Here are some deep results on different (=non-diffeomorphic) smooth structures on a given topological manifold:

- (T. Rado, E. Moise) There is only one smooth structure for any topological manifold of dimension smaller than 4.
- (J. Milnor and M. Kervaire) The topological 7-sphere admits exactly 28 different smooth structures. More generally, it is known that for spheres, the number of different smooth structures are (from Wikipedia)

Dimension	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Smooth types	1	1	1	≥1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16	523264	24

The very special case is S^4: we don't know whether the smooth structure on S^4 is unique. This is known as the *smooth Poincaré conjecture*.

- (S. Donaldson and M. Freedman) For any $n \neq 4$, $\mathbb{R}^n$ has a unique smooth structure up to diffeomorphism; but on $\mathbb{R}^4$ there exist uncountable many distinct pairwise non-diffeomorphic smooth structures.

On the other hand, it is also known that the obstruction of the existence/uniqueness of smooth structure lies only in topological manifold:

- (Whitney) Any C^1 manifold admits a unique smooth structure. (c.f. Theorem 2.9 in Hirsch, Differential Topology, GTM 33)

2. The differential of a smooth map: the Euclidean case

¶ The differentials of Euclidean smooth maps.

Let U, V be Euclidean open sets, and $f : U \to V$ be a smooth map. The *differential* of f, df, assigns to each point $a \in U$ a linear map $df_a : \mathbb{R}^n \to \mathbb{R}^m$ whose matrix (with respect to the canonical basis) is the *Jacobian matrix* of f at a,

$$df_a = \begin{pmatrix} \frac{\partial f_1}{\partial x^1}(a) & \cdots & \frac{\partial f_1}{\partial x^n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x^1}(a) & \cdots & \frac{\partial f_m}{\partial x^n}(a) \end{pmatrix}.$$

As we have seen in multi-variable calculus, the differential df_x plays a crucial role in studying the map f, since it is the "linearization" of the map f near the point x:

$$\lim_{x \to a} \frac{\|f(x) - f(a) - df_a(x-a)\|}{\|x - a\|} = 0.$$

A very useful fact for the differential is the well-known **chain rule**: if $f : U \to V$ and $g : V \to W$ are smooth maps, so is the composition map $g \circ f : U \to W$, and

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

One should regard d as a "functor" from the category of whose objects are "Euclidean open sets" and morphisms are "smooth maps to the category with objects "vector spaces" and morphisms "linear maps". Then the chain rule is merely part of the functor proprety. As a consequence of the use of the linear approximation, we can prove the following smooth "invariance of domain" theorem:

Theorem 2.1 (Invariance of Dimension, smooth version)**.** *If $f : U \to V$ is a diffeomorphism, then for each $x \in U$, the differential df_x is a linear isomorphism. In particular,* $\dim U = \dim V$.

Proof. Applying the chain rule to $f^{-1} \circ f = \mathrm{Id}_U$, and notice that the differential of the identity map $\mathrm{Id_U} : U \to U$ is the identity transformation $\mathrm{Id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$, we get

$$(df^{-1})_{f(x)} \circ df_x = \mathrm{Id}_{\mathbb{R}^n}.$$

The same argument applies to $f \circ f^{-1}$, which yields $df_x \circ (df^{-1})_{f(x)} = \mathrm{Id}_{\mathbb{R}^m}$. By basic linear algebra, we conclude that $m = n$ and that df_x is an isomorphism. □

¶ **Reading material: The inverse/implicit function theorems.**

Conversely, one would like to ask: if the linearization df_x is an linear isomorphism, what can we say about the map f itself? In general f is no longer a diffeomorphism:

Example. Consider the map

$$f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, \quad z \mapsto f(z) = z^2.$$

Then $f(z) = f(-z)$. So f is not a diffeomorphism since it is not invertible. However, at each point $z = (x, y) \in \mathbb{R}^2 \setminus \{0\}$,

$$df_z = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix},$$

which is an isomorphism for each $z = (x, y) \neq (0, 0)$. Fortunately, the situation is not that bad: f is not very far away from being a diffeomorphism: for any $x \in \mathbb{C} \setminus \{0\}$, one can find a small neighborhood U_x of x, such that the restriction $f|_{U_x} : U_x \to f(U_x)$ is a diffeomorphism. This motivates the following definition:

Definition 2.2. Let $f : U \to V$ be a smooth map. We say f is a *local diffeomorphism* near $x \in U$ if there exists a neighborhood U_x containing x and a neighborhood $V_{f(x)}$ containing $f(x)$ such that

$$f|_{U_x} : U_x \to V_{f(x)}$$

is a diffeomorphism

Now we can state the inverse function theorem, which can be viewed as a partial inverse of the "smooth invariance of domain" theorem.

Theorem 2.3 (The inverse function theorem)**.** *If $f : U \to V$ is a smooth map, and df_x is an isomorphism, then f is a* local diffeomorphism *near x.*

In other words, an isomorphism in the linear category implies a *local* diffeomorphism in the smooth category. The inverse function theorem is a special case (why) of the following implicit function theorem in multivariable analysis course:

Theorem 2.4 (The implicit function theorem)**.** *Let W be an open set in $\mathbb{R}^n_x \times \mathbb{R}^m_y$, and $F = (F_1, \cdots, F_m) : W \to \mathbb{R}^m$ a smooth map. Let (x_0, y_0) be a point in W so that the $m \times m$ matrix*

$$\begin{pmatrix} \frac{\partial F_1}{\partial y^1}(x_0, y_0) & \cdots & \frac{\partial F_1}{\partial y^m}(x_0, y_0) \\ \vdots & \vdots & \vdots \\ \frac{\partial F_m}{\partial y^m}(x_0, y_0) & \cdots & \frac{\partial F_m}{\partial y^m}(x_0, y_0) \end{pmatrix}.$$

is nonsingular, then there exists a neighborhood $U_0 \times V_0$ of (x_0, y_0) in W and a smooth map $f : U_0 \to V_0$ so that the graph of f is the level set of F near (x_0, y_0), i.e.

- $f(x_0) = y_0$,
- *If we denote $c = F(x_0, y_0)$, then $F^{-1}(c) \cap (U_0 \times V_0)$ is the graph of f, i.e. $F(x, f(x)) = c$ for all $x \in U_0$.*