LECTURE 6: LOCAL BEHAVIOR VIA THE DIFFERENTIAL

1. The Inverse function theorem

¶ The inverse function theorem.

Last time we showed that if $f : M \to N$ is a diffeomorphism, then the differential $df_p : T_pM \to T_{f(p)}N$ is a linear isomorphism. As in the Euclidean case (see Lecture 4), the converse is not true in general (i.e., "$f : M \to N$ is smooth and df_p is a linear isomorphism for every $p \in M$" does not imply "f is a diffeomorphism") but we still have the following partial converse:

Theorem 1.1 (The Inverse Function Theorem)**.** *Let $f : M \to N$ be a smooth map such that $df_p : T_pM \to T_{f(p)}N$ is a linear isomorphism, then there exists a neighborhood U_1 of p and a neighborhood X_1 of $q = f(p)$ such that $f|_{U_1} : U_1 \to X_1$ is a diffeomorphism.*

Proof. Take a chart (φ, U, V) near p and a chart (ψ, X, Y) near $f(p)$ so that $f(U) \subset X$ (which is always possible after shrinking U and V). Since $\varphi : U \to V$ and $\psi : X \to Y$ are diffeomorphisms,

$$d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} = d\psi_q \circ df_p \circ d\varphi^{-1}_{\varphi(p)} : T_{\varphi(p)}V = \mathbb{R}^n \to T_{\psi(q)}Y = \mathbb{R}^n$$

is a linear isomorphism. It follows from the inverse function theorem (c.f. Lecture 4) that there exist neighborhoods V_1 of $\varphi(p)$ and Y_1 of $\psi(q)$ so that $\psi \circ f \circ \varphi^{-1}$ is a diffeomorphism from V_1 to Y_1. Take $U_1 = \varphi^{-1}(V_1)$ and $X_1 = \psi^{-1}(Y_1)$. Then

$$f = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi$$

is a diffeomorphism from U_1 to X_1. □

¶ Local diffeomorphism v.s. global diffeomorphism.

Definition 1.2. We say a smooth map $f : M \to N$ is a *local diffeomorphism* near p, if it maps an open neighborhood of p diffeomorphically to an open neighborhood of $f(p)$.

Note that it is possible that a map is a local diffeomorphism everywhere, but still fails to be diffeomorphism.

Example. Let $f : S^1 \to S^1$ be given by $f(e^{i\theta}) = e^{2i\theta}$. Then it is a local diffeomorphism everywhere, but it is not a global diffeomorphism since it is not invertible. [Please compare this example with the example on page 5 of Lecture 4.]

It turns out that the invertibility is the only obstruction for an "everywhere local diffeomorphism" to be a global diffeomorphism:

Proposition 1.3. *Suppose $f : M \to N$ is a local diffeomorphism near every $p \in M$. If f is invertible, then f is a global diffeomorphism.*

Proof. We only need to show f^{-1} is smooth. Fix any $q = f(p)$. The smoothness of f^{-1} at q depends only on the behaviour of f^{-1} near q. Since f is a diffeomorphism from a neighborhood of p onto a neighborhood of q, f^{-1} is smooth at q. □

2. The constant rank theorem

¶ Submersion and immersion.

What if df_p is not a linear isomorphism? Note that a linear isomorphism is both surjective and injective. It is natural to study the those smooth maps whose differential is either surjective or injective:

Definition 2.1. Let $f : M \to N$ be a smooth map.

(1) f is a *submersion* at p if $df_p : T_pM \to T_{f(p)}N$ is surjective.
(2) f is an *immersion* at p if $df_p : T_pM \to T_{f(p)}N$ is injective.

We say f is a submersion/immersion if it is a submersion/immersion at each point.

Obviously

- If f is a submersion, then $\dim M \geq \dim N$.
- If f is an immersion, then $\dim M \leq \dim N$.

Example. In PSet 2-1-6, we see: the natural projection $\pi : TM \to M$ is a submersion. Similarly, the "zero section" embedding $\iota : M \to TM, p \mapsto (p, 0)$ is an immersion.

Example. A local diffeomorphism is both a submersion and an immersion.

Example (Canonical submersion)*.* If $m \geq n$, then the projection map

$$\pi : \mathbb{R}^m \to \mathbb{R}^n, \quad (x^1, \cdots, x^m) \mapsto (x^1, \cdots, x^n)$$

is a submersion.

Example (Canonical immersion)*.* If $m \leq n$, then the inclusion map

$$\iota : \mathbb{R}^m \hookrightarrow \mathbb{R}^n, \quad (x^1, \cdots, x^m) \mapsto (x^1, \cdots, x^m, 0, \cdots, 0)$$

is an immersion.

It turns out that any submersion/immersion locally looks like these two canonical ones.

Theorem 2.2 (Canonical Submersion Theorem)**.** *Let $f : M \to N$ be a submersion at $p \in M$, then $m = \dim M \geq n = \dim N$, and there exist charts (φ_1, U_1, V_1) around p and (ψ_1, X_1, Y_1) around $q = f(p)$ such that*

$$\psi_1 \circ f \circ \varphi_1^{-1} = \pi|_{V_1}.$$

Theorem 2.3 (Canonical Immersion Theorem)**.** *Let $f : M \to N$ be an immersion at $p \in M$, then $m = \dim M \leq n = \dim N$, and there exist charts (φ_1, U_1, V_1) around p and (ψ_1, X_1, Y_1) around $q = f(p)$ such that*

$$\psi_1 \circ f \circ \varphi_1^{-1} = \iota|_{V_1}.$$

¶ The constant rank theorem.

We will not prove the canonical submersion/immersion theorems above. Instead, we will prove a more general theorem which has the canonical submersion/immersion theorems as special cases. For this purpose, we define

Definition 2.4. We say a smooth map $f : M \to N$ is a *constant rank map* near $p \in M$ if there is a neighborhood U of p so that df_q has constant rank (i.e. there exists $r \in \mathbb{N}$ so that $\mathrm{rank}(df)_q \equiv r$) for all $q \in U$.

Example. If f is a submersion/immersion at p, then it is a submersion/immersion near p (why?), and thus is a constant rank map near p.

Example ("Canonical" constant rank map). More generally, by composing suitable canonical submersion and canonical immersion, we get a constant rank map

$$\mathbb{R}^m = \mathbb{R}^{r+m-r} \xrightarrow{\pi} \mathbb{R}^r \xrightarrow{\iota} \mathbb{R}^{r+n-r} = \mathbb{R}^n$$

which sends $(x^1, \cdots, x^r, x^{r+1}, \cdots, x^m) \in \mathbb{R}^m$ to $(x^1, \cdots, x^r, 0, \cdots, 0) \in \mathbb{R}^n$.

We shall prove:

Theorem 2.5 (The Constant Rank Theorem)**.** *Let $f : M \to N$ be a smooth map so that* $\mathrm{rank}(df) \equiv r$ *near p. Then there exists charts (φ_1, U_1, V_1) around p and (ψ_1, X_1, Y_1) near $f(p)$ such that that*

$$\psi_1 \circ f \circ \varphi_1^{-1}(x^1, \cdots x^m) = (x^1, \cdots, x^r, 0, \cdots, 0).$$

Proof. As usual we will convert the general case to the Euclidian case.

Step 1: The Euclidean case.

We first assume $U \subset \mathbb{R}^m$ is open, and $f : U \to \mathbb{R}^n$ is a smooth map so that df_x has constant rank r for all $x \in U$. By translation (in both $\mathbb{R}^m$ and $\mathbb{R}^n$, which amounts to composing f with suitable "translation diffeomorphisms" in both sides) we may assume $0 \in U$ and $f(0) = 0$. Since $\mathrm{rank}(df)_0 = r$, by switching coordinates (again in both $\mathbb{R}^m$ and $\mathbb{R}^n$, which amounts to composing f with suitable "switching coordinates diffeomorphisms" in both sides) we may assume that the upper-left $r \times r$ submatrix,

$$\left(\frac{\partial f_i}{\partial x^j}\right)_{1\le i,j\le r},$$

of the Jacobian $df = (\frac{\partial f_i}{\partial x^j})_{1\le i\le n, 1\le j\le m}$ is nonsingular at $x = 0$ (and thus is nonsingular near $x = 0$).

[The idea: Since $\mathrm{rank}\left(\frac{\partial f_i}{\partial x^j}\right)_{1\le i,j\le r} = \mathrm{rank}(\frac{\partial f_i}{\partial x^j})_{1\le i\le n,1\le j\le m}$, we may try to take $f_1, \cdots, f_r$ as part of our coordinates, so that with respect to these new coordinates, f will keep the first r coordinates unchanged.] Now define $\varphi : U \to \mathbb{R}^m$ by

$$\varphi(x) = (f_1(x), \cdots, f_r(x), x^{r+1}, \cdots, x^m).$$

Then $\varphi(0) = 0$, and the differential

$$d\varphi = \begin{pmatrix} \left(\frac{\partial f_i}{\partial x^j}\right)_{1\le i,j\le r} & * \\ 0 & \mathrm{Id}_{n-r} \end{pmatrix}$$

is nonsingular at $x=0$. By the inverse function theorem, φ is a local diffeomorphism near 0, i.e., there exists neighborhood U_1 of 0 in $\mathbb{R}^m$ and V_1 of 0 in $\mathbb{R}^m$ such that $\varphi : U_1 \to V_1$ is a diffeomorphism. Note that by definition,

$$f \circ \varphi^{-1}(f_1(x), \cdots, f_r(x), x^{r+1}, \cdots, x^m) = f \circ \varphi^{-1}(\varphi(x)) = f(x) = (f_1(x), \cdots, f_n(x))$$

i.e., locally near 0 we have

$$f \circ \varphi^{-1}(x) = (x^1, \cdots, x^r, g_{r+1}(x), \cdots, g_n(x))$$

for some smooth functions $g_{r+1}, \cdots, g_n$ (with $g_i(0)=0$). Moreover, by chain rule,

$$df_{\varphi^{-1}(x)} \circ (d\varphi^{-1})_x = \begin{pmatrix} \mathrm{Id}_r & 0 \\ * & \left(\frac{\partial g_i}{\partial x^j}\right)_{r+1 \le i \le n, r+1 \le j \le m} \end{pmatrix}.$$

Crucial observation: Since $(d\varphi^{-1})_x$ is a linear isomorphism, "rank$(df_x) = r$ near 0" implies "rank$(df_{\varphi^{-1}(x)} \circ (d\varphi^{-1})_x) = r$ near 0", and thus implies

$$\frac{\partial g_i}{\partial x^j} = 0, \qquad \forall r+1 \le i \le n, r+1 \le j \le m$$

near 0. It follows that in a small neighborhood of 0, we have

$$g_i(x) = g_i(x^1, \cdots, x^r), \qquad \forall r+1 \le i \le n.$$

In other words, near 0 we have

$$f \circ \varphi^{-1}(x) = (x^1, \cdots, x^r, g_{r+1}(x^1, \cdots, x^r), \cdots, g_n(x^1, \cdots, x^r)).$$

It remains to kill these g_i's. So we define

$$\psi(y) = \big(y^1, \cdots, y^r, y^{r+1} - g_{r+1}(y^1, \cdots, y^r), \cdots, y^n - g_n(y^1, \cdots, y^r)\big).$$

in a small neighborhood of 0, and get

$$\psi \circ f \circ \varphi^{-1}(x^1, \cdots, x^r, x^{r+1}, \cdots, x^n) = (x^1, \cdots, x^r, 0, \cdots, 0).$$

It remains to check that ψ is a local diffeomorphism. Again this follows from the inverse function theorem and a simple computation $d\psi_0 = \begin{pmatrix} \mathrm{Id}_r & 0 \\ * & \mathrm{Id}_{n-r} \end{pmatrix}$.

Step 2: The general case.

The general case follows easily (by the standard trick): Take a coordinate neighborhood (φ, U, V) near p and (ψ, X, Y) near $f(p)$, so that $f(U) \subset X$, and df_q has constant rank r on U. Then $\psi \circ f \circ \varphi^{-1} : V \to Y$ has constant rank r since

$$d(\psi \circ f \circ \varphi^{-1})_x = d\psi_{f(\varphi^{-1}(x))} \circ df_{\varphi^{-1}(x)} \circ (d\varphi^{-1})_x$$

and since $(d\varphi^{-1})_x$, $d\psi_{f(\varphi^{-1}(x))}$ are linear isomorphisms. Now the desired conclusion follows from the Euclidean case. □

As a consequence, we see a map is a constant rank map if and only if it can be written, locally, as a composition $j \circ s$, where s is a submersion while j is an immersion. In particular,

- If a constant rank map is surjective, then it is a submersion.
- If a constant rank map is injective, then it is an immersion.