LECTURE 8: SMOOTH SUBMANIFOLDS

1. Smooth submanifolds

¶ Smooth submanifolds: The definition.

Let $M$ be a smooth manifold of dimension $n$. What object can be called a “smooth submanifold” of $M$? [Recall: what is a vector subspace $W$ of a vector space $V$? $W$ should satisfy three conditions:

- $W$ is a subset of $V$;
- $W$ is a vector space by itself;
- the vector space structure on $W$ should be the restriction of the vector space structure on $V$.

Similarly, what is a subgroup of a group? What is a topological subspace of a topological space? In each case you can always write down the three conditions: the inclusion relation, the structure itself, and the compatibility.] A smooth submanifold $S$ of $M$ should be something that satisfies three the following conditions:

- $S$ should be a subset of $M$;
- $S$ itself should be a smooth manifold of dimension $k \leq n$;
- the smooth structures on $S$ and on $M$ should be compatible.

The last condition, i.e. the compatibility, can be stated more precisely: the smooth structure (={a set of coordinate charts}) on $S$ should be the “restriction” of the smooth structure (={a set of coordinate charts}) on $M$.

**Definition 1.1.** A subset $S \subset M$ is a $k$-dimensional smooth submanifold of $M$ if for every $p \in S$, there is a chart $(\varphi, U, V)$ around $p$ of $M$ such that

$$\varphi(U \cap S) = V \cap (\mathbb{R}^k \times \{0\}) = \{x \in \varphi(U) \mid x^{k+1} = \cdots = x^n = 0\}.$$ 

We will call $\text{codim}(S) = n - k$ the codimension of $S$.

![Diagram](image.png)

**Remark.** Roughly speaking, smooth submanifolds are objects that are defined locally (on a coordinate chart) by equations

$$\varphi_{k+1} = \cdots = \varphi_n = 0.$$ 

Note that $\varphi_{k+1}, \cdots, \varphi_n$ are smooth functions on $U$, since $\varphi$ is a diffeomorphism.
Smooth submanifolds: Examples.

Example. Let $M, N$ be smooth manifolds, and $f: M \to N$ be smooth. Then the graph
$$
\Gamma_f = \{(p, q) \mid q = f(p)\} \subset M \times N
$$
is a smooth submanifold of $M \times N$. To see this, we take a chart $(\varphi, U, V)$ of $M$ near $p$ and a chart $(\psi, X, Y)$ of $N$ near $q = f(p)$. Then (c.f. PSet 1-1-6) $(\varphi \times \psi, U \times X, V \times Y)$ is a chart of $M \times N$ near $(p, q)$. But this chart is not good for our purpose. To get a chart that is suitable for our purpose, we write the equation $q = f(p)$ as $\psi^{-1}(y) = f(\varphi^{-1}(x))$, i.e. $y = \psi(f(\varphi^{-1}(x)))$. Now we define a smooth map
$$
\Psi: V \times Y \to \mathbb{R}^m \times \mathbb{R}^n, \quad (a, b) \mapsto (a, b - \psi \circ f \circ \varphi^{-1}(a)).
$$
It is easy to see that $\Psi$ is one-to-one and is a local diffeomorphism everywhere, thus is a global diffeomorphism from $V \times Y$ onto its image $\Psi(V \times Y)$, which is an Euclidian open set. Thus $(\Psi \circ (\varphi \times \psi), U \times X, \Psi(V \times Y))$ is also a local chart of $M \times N$ near $(p, q)$. Moreover, with respect to this local chart,
$$
(p, q) \in \Gamma_f \cap (U \times X) \implies \psi(q) = \psi(f(\varphi^{-1}(\varphi(p)))) \implies \Psi(\varphi(p), \psi(q)) = (\varphi(p), 0),
$$
so the conclusion follows.

Remark. In Lecture 2 we mentioned that for any continuous function $f: U \subset \mathbb{R}^n \to \mathbb{R}$, there is a smooth structure on the graph $\Gamma_f$ to make it a smooth manifold of dimension $n$. However, in general $\Gamma_f$ is not a smooth submanifold of $\mathbb{R}^{n+1}$ if $f$ is not smooth.

- If you repeat the arguments in the above example, which step does not work?
- However, it is possible $f$ is not a smooth function, while $\Gamma_f$ is still a smooth submanifold: Consider the function $f(x) = x^{1/3}$. Then the graph of $y = f(x)$ is the same as the graph of the function $x = y^3$, which is of course a smooth submanifold of $\mathbb{R}^2$!

Example. The sphere $S^n$ is a smooth submanifold of $\mathbb{R}^{n+1}$. Can you construct a local chart of $\mathbb{R}^{n+1}$ near every point of $S^n$ which satisfies the condition in the definition 1.1?

The induced smooth structures on smooth submanifolds.

Note that in the definition of a smooth submanifold above, we did not spell out the smooth structure on $S$. To construct natural charts on $S$, we denote
$$
\pi: \mathbb{R}^n \to \mathbb{R}^k, \quad (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^k)
$$
and
$$
j: \mathbb{R}^k \hookrightarrow \mathbb{R}^n, \quad (x^1, \ldots, x^k) \mapsto (x^1, \ldots, x^k, 0, \ldots, 0).
$$
Then we have

Proposition 1.2. Let $(\varphi, U, V)$ be a chart on $M$ that satisfies Definition 1.1. Let $X = U \cap S$, $Y = \pi \circ \varphi(X)$ and $\psi = \pi \circ \varphi|_X$. Then $(\psi, X, Y)$ is a smooth chart on $S$ and charts of this form are compatible, so that $S$ is a smooth manifold. Moreover, the inclusion map $i: S \hookrightarrow X$ is a smooth immersion.

Proof. By definition, $\psi$ is invertible and the inverse $\psi^{-1} = \varphi^{-1} \circ j$. So $(\psi, X, Y)$ is a chart on $S$. It remains to check that charts of this type are compatible. In fact, the transition maps are
Smooth submanifolds as level sets.

\[ \psi_\beta \circ \psi_\alpha^{-1} = \pi \circ \varphi_\beta \circ \varphi_\alpha^{-1} \circ j = \pi \circ \varphi_{\alpha,\beta} \circ j, \]

which are smooth. Moreover, with respect to these smooth structures, the inclusion map \( \iota : S \hookrightarrow M \) is a smooth immersion since by definition,

\[ \varphi \circ \iota \circ \psi^{-1} = j. \]

\[ \square \]

The tangent space of a smooth submanifold (as a vector subspace).

Now let \( S \subset M \) be a submanifold, and \( p \in S \). Since \( \iota : S \hookrightarrow M \) is an embedding, \( d\iota_p : T_pS \to T_pM \) is injective. We might identify \( T_pS \) with the vector subspace \( d\iota_p(T_pS) \) of \( T_pM \) for every \( p \in S \). In other words, we can identify any vector \( X_p \in T_pS \) with the vector \( \tilde{X}_p = d\iota_p(X_p) \) in \( T_pM \) so that for any \( f \in C^\infty(M) \),

\[ \tilde{X}_p(f) = (d\iota_p(X_p))f = X_p(f \circ \iota) = X_p(f|_S). \]

A natural question is: which vectors in \( T_pM \) can be regarded as vectors in \( T_pS \)?

**Theorem 1.3.** Suppose \( S \subset M \) is a submanifold, and \( p \in S \). Then

\[ T_pS = \{ X_p \in T_pM \mid X_p(f) = 0 \text{ for all } f \in C^\infty(M) \text{ with } f|_S = 0 \}. \]

**Proof.** From the description above, we see if \( X_p \in T_pS \), then for \( f \in C^\infty(M) \) with \( f|_S = 0 \), \( \tilde{X}_p(f) = X_p(f|_S) = 0 \).

Conversely, if \( X_p \in T_pM \) satisfies \( X_p(f) = 0 \) for all \( f \) that vanishes on \( S \), we need to show \( X_p \in T_pS \). Take a coordinate chart \((\varphi, U, V)\) on \( M \) such that near \( p \), \( S \) is given by \( x^{k+1} = \cdots = x^n = 0 \). Then \( T_pM \) is the span of \( \partial_1, \ldots, \partial_n \), while \( T_pS \) is the subspace spanned by \( \partial_1, \ldots, \partial_k \). In other words, a vector \( X_p = \sum X^i \partial_i \) lies in \( T_pS \) if and only if \( X^i = 0 \) for \( i > k \).

Now let \( h \) be a smooth bump function supported in \( U \) that equals 1 in a neighborhood of \( p \). For any \( j > k \), consider the function \( f_j(x) = h(x)x^j(\varphi(x)) \), extended to be zero on \( M \setminus U \). Then \( f_j|_S = 0 \). So

\[ 0 = X_p(f_j) = \sum X^i \frac{\partial(h(\varphi^{-1}(x))x^j)}{\partial x^i}(\varphi(p)) = X^j \]

for any \( j > k \). It follows that \( X_p \in T_pS \).

\[ \square \]

2. Smooth submanifolds as pre-images and images

Smooth submanifolds as level sets.

Now we prove the result that we mentioned last time: the pre-image of a regular value of any smooth map is a smooth submanifold. [Thus, as a consequence of Sard’s theorem, for any smooth map \( f : M \to N \), if \( f(M) \) is not negligible in \( N \), then for most \( q \in f(M) \), the pre-image \( f^{-1}(q) \) is a smooth submanifold of \( M \).]

**Theorem 2.1** (Regular Level Set Theorem). Let \( f : M \to N \) be a smooth map and \( q \in N \) a regular value of \( f \). Then \( f^{-1}(q) \) is a smooth submanifold of \( M \) of dimension \( \dim M - \dim N \). Moreover, for every \( p \in S \), \( T_pS = \ker(df_p : T_pM \to T_qN) \).
Note that if \( q \) is a regular value, then \( f \) is a full rank map near any \( p \in f^{-1}(q) \). So the regular level set theorem is a consequence of

**Theorem 2.2 (Constant rank level set theorem).** Let \( M, N \) be smooth manifolds, and \( f : M \to N \) be a smooth map with constant rank \( r \). Then each level set of \( f \) is a closed submanifold of codimension \( r \) in \( M \). Moreover, for every \( p \in S \), \( T_p S \) (viewed as a vector subspace of \( T_p M \)) is the kernel of the map \( df_p : T_p M \to T_q N \).

**Proof.** Let \( p \in S := f^{-1}(q) \). Then by the constant rank theorem that we proved in Lecture 6, there are charts \((\varphi_1, U_1, V_1)\) centered at \( p \) and \((\psi_1, X_1, Y_1)\) centered at \( q \) such that \( f(U_1) \subset X_1 \), and

\[
\psi_1 \circ f \circ \varphi_1^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).
\]

It follows that \( \varphi_1 \) maps \( U_1 \cap f^{-1}(q) \) onto \( V_1 \cap \{(0, \ldots, 0, x^{r+1}, \ldots, x^m)\} \). So \( f^{-1}(q) \) is a submanifold of \( M \) of codimension \( r \).

Next denote the inclusion by \( \iota : S \to M \). Then for any \( p \in S \), \( f \circ \iota(p) = q \). In other words, \( f \circ \iota \) is a constant map on \( S \). So \( df_p \circ dt_p = 0 \), i.e. \( df_p = 0 \) on the image of \( dt_p : T_p S \hookrightarrow T_p M \), or in other words, \( T_p S \subset \ker(df_p) \). But \( \dim T_p S = \dim S = m - r \), and

\[
\dim \ker(df_p) = \dim \ker((d\psi_1)_{f(p)} \circ df_p \circ (d\varphi_1^{-1})_{\varphi(p)}) = m - r.
\]

By dimension counting we conclude that \( T_p S \) coincides with the kernel of \( df_p \). \( \square \)

The regular level set theorem and the constant rank theorem are very powerful tools in applications. For example, we immediately see

- \( S^n \) is a smooth submanifold of \( \mathbb{R}^{n+1} \).
- According to PSet2-2-4(a), \( \text{SL}(n, \mathbb{R}) \) is a smooth submanifold of \( \text{GL}(n, \mathbb{R}) \).
- According to PSet2-2-4(b), \( \text{O}(n, \mathbb{R}) \) is a smooth submanifold of \( \text{GL}(n, \mathbb{R}) \).

Note that the level set of a critical value may fail to be a smooth manifold. For example, consider the function

\[
f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = x^2 - y^2.
\]

Then \( df(x, y) = (2x, 2y) \) and thus the only critical point is \((0, 0)\). For the only critical value 0, the level set \( f^{-1}(0) \) is not a manifold.

**Smooth submanifolds as images: immersions are not enough.**

As we have seen, any smooth submanifold \( S \) of \( M \) is the image of a smooth immersion \( \iota : S \hookrightarrow M \). It is natural to ask: whether the image of any smooth immersion is a smooth submanifold? First by the canonical immersion theorem, if \( f : N \to M \) is an immersion, then for each \( p \in N \), there exists a coordinate neighborhood \( U \) of \( p \) so that \( f(U) \) is a smooth submanifold of \( M \) (why? write down a proof). Unfortunately in general \( f(N) \) is need not be a smooth submanifold of \( M \):

**Example.** The following two graphs are the images of two immersions of \( \mathbb{R} \) into \( \mathbb{R}^2 \). For the first one, the immersion is not injective. For the second one, the immersion is injective, while the image still have different topology than \( \mathbb{R} \).
Example. There is a more complicated/interesting example: consider the map \( f : \mathbb{R} \to T^2 = S^1 \times S^1 \), \( f(t) = (e^{it}, e^{i\sqrt{2}t}) \). Then \( f \) is an immersion (why?), and the image \( f(\mathbb{R}) \) is a “dense curve”\(^1\) in \( T^2 \).

Remark. For the three immersions above whose image are not submanifolds, the first one is “worst” since the image is not a manifold in any sense: at the crossing point, the image is not a manifold, no matter what topology you give to the image. On the other hand, for the second one and the third one, we can easily see that

- if we use the “subspace topology” inherited from \( \mathbb{R}^2 \) or \( T^2 \), then the images are not manifolds;
- if we endow the images with the topology that “borrowed” from \( \mathbb{R} \), then the images are smooth manifolds by themselves!

In general, the image of any injective immersion is a manifold, where the manifold structure is “borrowed” from the source manifold. So people call the images of injective immersions immersed submanifolds. To distinguish immersed submanifolds with smooth submanifolds defined in Definition 1.1, sometimes people call smooth submanifolds embedded submanifolds or regular submanifolds.

¶ Smooth submanifolds as images: embeddings.

What is the difference between smooth submanifolds and immersed submanifolds? As we just described, the topology that makes an immersed submanifold a manifold is the topology from the source manifold, not the “subspace topology” from the target manifold. On the other hand, if \( S \) is a smooth submanifold of \( M \), then the underlying topology of the smooth manifold \( S \) is the topology generated by charts \((\psi, X, Y)\) in Proposition 1.2. By definition 1.1 it is easy to see

\(^1\)This is a consequence of the fact that \( \{\{n\sqrt{2}\} \mid n \in \mathbb{Z}\} \) is dense in \([0, 1]\).
Proposition 2.3. Let $S$ be a smooth submanifold of $M$. Then $\iota : S \hookrightarrow M$ is a homeomorphism from $S$ to $\iota(S)$ (endowed with the subspace topology from $M$).

Remark. If $S$ is a smooth submanifold of $M$, then there is a unique topology/smooth structure on $S$ so that the inclusion map $\iota : S \hookrightarrow M$ is a smooth immersion which is a homeomorphism onto its image. (See Theorem 5.31 in Lee’s book.)

So for any smooth submanifold $S$, the inclusion map $\iota : S \hookrightarrow M$ is a special immersion which is a homeomorphism onto its image.

Definition 2.4. Let $M, N$ be smooth manifolds, and $f : N \rightarrow M$ an immersion. $f$ is called an embedding if it is a homeomorphism onto its image $f(N)$, where the topology on $f(N)$ is the subspace topology as a subset of $M$.

By definition, the inclusion map $\iota : S \hookrightarrow M$ is an embedding. So each smooth submanifold is the image of an embedding.

Conversely, Theorem 2.5. Let $f : N \rightarrow M$ be an embedding. Then the image $f(N)$ is a smooth submanifold of $M$.

Proof. Let $p \in N$ and $q = f(p)$. Since $f$ is an immersion, the canonical immersion theorem implies that there exists charts $(\varphi_1, U_1, V_1)$ near $p$ and $(\psi_1, X_1, Y_1)$ near $q$ such that on $V_1$, $\psi_1 \circ f \circ \varphi_1^{-1}$ is the canonical embedding $\iota : \mathbb{R}^m \rightarrow \mathbb{R}^n$ restricted to $V_1$, i.e.

$$\psi_1 \circ f = \iota \circ \varphi_1$$

on $U_1$. Since $f$ is a homeomorphism onto its image, $f(U_1)$ is relative open in the subspace $f(N) \subset M$. In other words, there exists an open set $X \subset M$ such that $f(U_1) = f(N) \cap X$. Replace $X_1$ by $X_1 \cap X$, and $Y_1$ by $\psi_1(X_1 \cap X)$. Then for this new chart $(\psi_1, X_1, Y_1)$,

$$\psi_1(X_1 \cap f(N)) = Y_1 \cap \psi_1(f(U_1)) = Y_1 \cap \iota(\varphi_1(U_1)) = Y_1 \cap (\mathbb{R}^m \times \{0\}).$$

[Please repeat the above argument for the “dense curve in $\mathbb{T}^2$” example to see what’s wrong there.]

Let’s summarize the main difference between an immersion and an embedding:

- If $f : N \rightarrow M$ is an immersion, then by the canonical immersion theorem, any point $p \in N$ has a neighborhood in $N$ whose image is “nice” in $M$.
- If $f : N \rightarrow M$ is an embedding, then by Theorem 2.5, any point $q \in f(N)$ has a neighborhood in $f(N)$ that is “nice” in $M$.