LECTURE 10: TUBULAR NEIGHBORHOOD THEOREM

1. Generalized Inverse Function Theorem

¶ Generalized Inverse Function Theorem.

In Lecture 6 we proved the inverse function theorem which claims that if df_p is a linear isomorphism, then f is a local diffeomorphism near p, i.e. f is maps a neighborhood of p diffeomorphically to a neighborhood of $f(p)$. In applications, one may need to map a neighborhood of a subset/submanifold X diffeomorphically to a neighborhood of $f(X)$. It turns out that under suitable assumptions, the inverse function theorem can be extended to this setting to produce such a diffeomorphism on a neighborhood a smooth submanifold:

Theorem 1.1 (Generalized Inverse Function Theorem)**.**
Let $f : M \to N$ be a smooth map, and $X \subset M$ a submanifold. Suppose $df_x : T_xM \to T_{f(x)}N$ is a linear isomorphism for each $x \in X$. Moreover,

Case 1: If X is compact, we simply assume f is injective on X.

Case 2: If X is non-compact, we assume $f(X)$ is a submanifold of N and f maps X diffeomorphically onto $f(X)$.

Then f maps a neighborhood U of X in M diffeomorphically onto a neighborhood V of $f(X)$ in N.

We give a couple remarks on the conditions:

Remark. In fact, the simple condition "f is injective on X" is equivalent to the complicated condition "X diffeomorphically onto $f(X)$" when X is compact: By assumption, $f|_X : X \to N$ is an immersion. So if $f|_X$ is injective and X is compact, then $f|_X : X \to N$ is an embedding, and thus $f|_X : X \to f(X)$ is a diffeomorphism.

Remark. When X is non-compact, the theorem may fail if we only assume X is a smooth submanifold and $f|_X$ is injective. Here is a simple example:

Consider the covering map

$$f : \mathbb{R}^2 \to \mathbb{T}^2, \quad (t, s) \mapsto (e^{it}, e^{is})$$

which is a local diffeomorphism near any $(t, s) \in \mathbb{R}^2$. We take X to be the "irrational-slope line"

$$X = \{(t, \sqrt{2}t) \mid t \in \mathbb{R}\}$$

in $\mathbb{R}^2$ (which is a perfectly nice smooth submanifold), then $f|_X$ is injective but there is no hope to find neighborhoods U of X in $\mathbb{R}^2$ and V of $f(X)$ in $\mathbb{T}^2$ so that f maps U diffeomorphically onto V.(why?)

¶ Generalized Inverse Function Theorem: The compact case.

Proof of Theorem 1.1, Case 1 (compact version).
By the inverse function theorem, f is a local diffeomorphism near each point in X. According to Proposition 1.3 in Lecture 6, it is enough to prove that f is injective on a neighborhood of X. For this purpose we embed M into $\mathbb{R}^K$, and consider the "ε-neighborhood" of X in M:

$$X^\varepsilon = \{x \in M \mid d(x, X) < \varepsilon\},$$

where $d(\cdot,\cdot)$ is the Euclidean distance between a point and a set, namely

$$d(x, X) = \inf\{d(x, y) \mid y \in X\}.$$

Note that X^ε is a open subset in $\mathbb{R}^K$, and it is bounded since X is compact. Moreover we have $X = \bigcap_{k>0} X^{1/k}$ since X is closed. (Why?)

Now we proceed by contradiction. If f is not one-to-one on each $X^{1/k}$, then one can find $a_k \neq b_k \in X^{1/k}$ such that $f(a_k) = f(b_k)$. Since all a_k's lie in a bounded closed set in $\mathbb{R}^K$, which is compact, one can find a subsequence such that $a_{k_i} \to a_\infty \in X$. Similarly there is a subsequence $b_{k_{i_j}} \to b_\infty \in X$. Since $f(a_\infty) = f(b_\infty)$, one must have $a_\infty = b_\infty$, since f is one-to-one on X. So in any neighborhood of a_∞, f is not one-to-one. But df_{a_∞} is linear isomorphism implies that f is a local diffeomorphism near a_∞, which is a contradiction. □

By staring at the proof, one can see that in the proof we don't need the submanifold structure of X, the only requirement is that X is a compact subset in M. So what we really proved is a result with weaker assumption:

Theorem 1.2 (Generalized Inverse Function Theorem, compact subset version)**.**
Let $f: M \to N$ be a smooth map that is injective on a compact subset $X \subset M$, and suppose $df_x : T_xM \to T_{f(x)}N$ is a linear diffeomorphism for each $x \in X$. Then f maps a neighborhood U of X in M diffeomorphically onto a neighborhood V of $f(X)$ in N.

¶ Generalized Inverse Function Theorem: the noncompact case.

Proof of Theorem 1.1, Case 2 (non-compact version).
[The idea: Any non-compact manifold can be written as a union of countably many "compact stripes"] By using any positive smooth exhaustion function g on $f(X)$, we may decompose $f(X) = \bigcup_{k=1}^\infty K_k$, where $K_k = g^{-1}([k, k+1])$. Since $f|_X : X \to f(X)$ is a diffeomorphism, each $J_k := f|_X^{-1}(K_k)$ is compact in X and give us a decomposition $X = \bigcup_{k=1}^\infty J_k$. In particular, by Theorem 1.2, there is an open neighborhood $\widetilde{U}_k$ of $J_{k-1} \cup J_k \cup J_{k+1}$ in M, such that f is a diffeomorphism on $\widetilde{U}_k$. Since $f(X)$ is a smooth submanifold in N, by embedding N into $\mathbb{R}^K$ and using the induced distance function, one can easily prove

$$d_k := \mathrm{dist}(K_k, \bigcup_{j>k+1} K_j) > 0.$$

(Here we need to use the compactness of K_k and use the fact that $f(X)$ is a smooth submanifold. Figure out the detail.)

Now we choose a decreasing sequence of positive numbers $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ such that $\varepsilon_k < d_k/2$ for each k. Note that this implies $K_l^{\varepsilon_l} \cap K_k^{\varepsilon_k} = \emptyset$ if $|l-k| > 1$. Define

$$U_k = \widetilde{U}_{k-1} \cap \widetilde{U}_k \cap \widetilde{U}_{k+1} \cap f^{-1}(K_k^{\varepsilon_k}).$$

Then U_k is an open neighborhood of J_k, $V_k = f(U_k)$ is an open neighborhood of K_k, and f maps U_k diffeomorphically onto V_k. Moreover, the definition of U_k implies $U_{k-1} \cup U_k \cup U_{k+1} \subset \widetilde{U}_k$.

We define $U = \bigcup_{k\geq 1} U_k$ and $V = \bigcup_{k \geq 1} V_k$. Then U is an open neighborhood of X in M, V is an open neighborhood of $f(X)$ in N, and $f : U \to V$ is a local diffeomorphism everywhere. It remains to show f is injective on U. Suppose $x, y \in U$ and $f(x) = f(y)$. Then there exists k such that $f(x) = f(y) \in V_k \subset K_k^{\varepsilon_k}$. It follows $x, y \in U_{k-1} \cup U_k \cup U_{k+1} \subset \widetilde{U}_k$. Since f is a diffeomorphism on $\widetilde{U}_k$, we conclude $x = y$. □

2. Tubular Neighborhood Theorem

Now let $X \subset M$ be a smooth submanifold. We want to study the followong problem: What does M looks like "near" X? The famous tubular neighborhood theorem claims that X always admits a "tubular" neighborhood inside M. Moreover, the tubular neighborhood looks like a neighborhood of X inside its "normal bundle". This gives some kind of "canonical form" of a neighborhood of any smooth submanifold.

¶ Tubular neighborhood when embedded in $\mathbb{R}^K$.

We first prove the following "extrinsic version" of the tubular neighborhood theorem, which gives us a nice neighborhood of any submanifold in $\mathbb{R}^K$:

Theorem 2.1 (ε-Neighborhood Theorem)**.** *Let $\iota : X \hookrightarrow \mathbb{R}^K$ be a smooth submanifold. Then there exists a continuous positive-valued function $\varepsilon : X \to \mathbb{R}^+$, such that if we let X^ε be the ε-neighborhood of X (which is open in $\mathbb{R}^K$),*

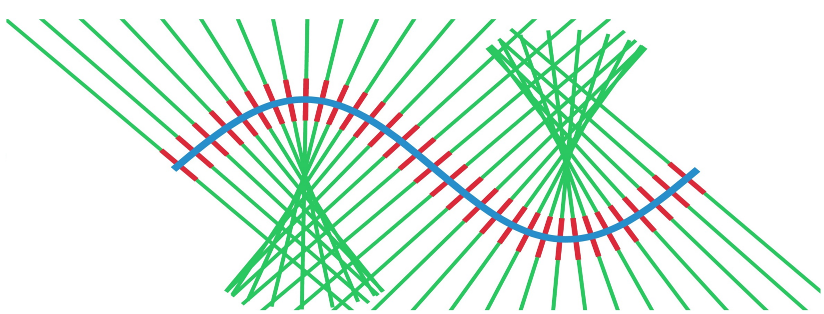
$$X^\varepsilon := \{y \in \mathbb{R}^K \mid |y - x| < \varepsilon(x) \text{ for some } x \in X\},$$

then

1. *each $y \in X^\varepsilon$ possesses a unique closest point $\pi_\varepsilon(y)$ in X;*
2. *the map $\pi_\epsilon : X^\varepsilon \to X$ is a submersion.*

(Note: If X is compact, then the function ε can be taken to be a constant.)

The ε-neighborhood described in the above theorem looks like



We will prove the theorem by constructing a diffeomorphism between a neighborhood of X inside its *normal bundle* (which can be think of as a nice "non-intersecting" way to put the green lines above together to form a smooth manifold) and a neighborhood of X inside $\mathbb{R}^K$. So let's start with the conception of the normal bundle of a manifold embedded in $\mathbb{R}^K$, with details left as an exercise:

Let $\iota : X \hookrightarrow \mathbb{R}^K$ be a smooth submanifold of dimension r. For each $x \in X$, we can identify T_xX with an r-dimensional vector subspace in $\mathbb{R}^K$ via

$$T_xX \simeq d\iota_x(T_xX) \subset T_x\mathbb{R}^K \simeq \mathbb{R}^K.$$

Let $N_x(X, \mathbb{R}^K)$ be the orthogonal complement of T_xX in $\mathbb{R}^K$,

$$N_x(X, \mathbb{R}^K) := \{v \in T_x\mathbb{R}^K \simeq \mathbb{R}^K \mid v \perp T_xX\},$$

which is a $(K - r)$-dimensional vector subspace of $\mathbb{R}^K$. Define

$$N(X, \mathbb{R}^K) = \{(x, v) \in \mathbb{R}^K \times \mathbb{R}^K \mid x \in X, v \in N_x(X, \mathbb{R}^K)\} \subset T\mathbb{R}^K.$$

We will leave it as an exercise for you to prove that $N(X, \mathbb{R}^K)$ is a K-dimensional smooth submanifold in $T\mathbb{R}^K$, and the canonical projection map

$$\pi : N(X, \mathbb{R}^K) \to X, \quad (x, v) \mapsto x$$

is a submersion.

Definition 2.2. Let $\iota : X \hookrightarrow \mathbb{R}^K$ be a smooth submanifold. We call $N(X, \mathbb{R}^K)$ described above *normal bundle* of X in $\mathbb{R}^K$.

Remark. The conception of normal bundle $N(X, \mathbb{R}^K)$ is extrinsic: it depends on the ambient space $\mathbb{R}^K$ and also depends on the way of embedding $\iota : X \hookrightarrow \mathbb{R}^K$.

Proof of the ε-Neighborhood Theorem.
Define a map

$$h : N(X, \mathbb{R}^K) \to \mathbb{R}^K, \quad (x, v) \mapsto x + v.$$

Then at any point $(x, 0) \in N(X, \mathbb{R}^K)$, dh is non-singular, because $dh_{(x,0)}$ maps $T_{(x,0)}(X \times \{0\}) \subset T_{(x,0)}N(X, \mathbb{R}^K)$ bijectively onto $T_xX \subset T_x\mathbb{R}^K$, and maps the tangent space $T_{(x,0)}(\{x\} \times N_x(X, \mathbb{R}^K))$ bijectively onto $N_x(X, \mathbb{R}^K) \subset T_x\mathbb{R}^K$.

Also by definition, h maps $X \times \{0\} \subset N(X, \mathbb{R}^K)$ diffeomorphically onto $X \subset \mathbb{R}^K$. According to the generalized inverse function theorem (GIFT) we just proved, h maps a neighborhood U of $X \times \{0\}$ in $N(X, \mathbb{R}^K)$ diffeomorphically onto a neighborhood V of X in $\mathbb{R}^K$.

Now for each $x \in X$, we define

$$\varepsilon(x) = \sup\{r \leq 1 \mid B_r(x) \subset V\}.$$

One can check that φ is a positive continuous function on X.(Check this!) Note that by definition, $X^\varepsilon \subset V$ is an open submanifold. Consider the map

$$\pi_\varepsilon : X^\varepsilon \to X, \qquad y \mapsto \pi_\varepsilon(y) = \pi \circ h^{-1}(y).$$

It is a submersion since π is a submersion and h^{-1} is a diffeomorphism on V. It remains to prove that $\pi_\varepsilon(y)$ is the unique closest point to y in X. In fact, let $z \in X$ be a point in X that is closest to y. Then the sphere centered at y with radius $|y - z|$ is tangent to X

at z. It follows that the vector $y-z$ is perpendicular to X at z, i.e. $y-z \in N_z(X, \mathbb{R}^K)$. So we have

$$y = z + (y-z) = h(z, y-z),$$

i.e. $\pi_\varepsilon(y) = z$. So the point z is unique, and $\pi_\varepsilon(y)$ is the unique point in X that is closest to y. This completes the proof. □

¶ The tubular neighborhood theorem.

In general, suppose $X \subset M$ is a smooth submanifold. Then one can still define the normal bundle $N(X, M)$ (with respect to the ambient space M) as the set of points of the form (x, v), where for any $x \in X$, v is any vector in the quotient space

$$N_x(X, M) = T_xM/T_xX.$$

One can show that $N(X, M)$ is a smooth manifold whose dimension equals $\dim M$. .

To get a more extrinsic (and geometric) description of $N(X, M)$, we may embed M into $\mathbb{R}^K$. Then we will get an inclusion $T_xX \subset T_xM \subset T_x\mathbb{R}^K$, and the quotient space T_xM/T_xX can be identified as the space of vectors in T_xM which are perpendicular to T_xX. It follows

$$N(X, M) \simeq \{(x, v) \mid x \in X, v \in T_xM \text{ and } v \perp T_xX\}.$$

Note that by using this identification, we have

$$T_{(x,0)}N(X, M) \simeq T_xX \oplus T_x^\perp X,$$

where $T_x^\perp X$ is the orthogonal complement of T_xX inside T_xM.

Now we prove

Theorem 2.3 (Tubular Neighborhood Theorem)**.** *Let $X \subset M$ be a smooth submanifold. Then there exists a diffeomorphism from an open neighborhood of X in $N(X, M)$ onto an open neighborhood of X in M.*

Proof. Embed M into $\mathbb{R}^K$. Let $\pi_\varepsilon : M^\varepsilon \to M$ be as in the ε-neighborhood theorem (for the embedding $\iota : M \hookrightarrow \mathbb{R}^K$). Again consider the map

$$h : N(X, M) \to \mathbb{R}^K, \quad h(x, v) \to x + v.$$

Then $W := h^{-1}(M^\varepsilon)$ is an open neighborhood of X in $N(X, M)$. Now consider the composition

$$h_\varepsilon = \pi_\varepsilon \circ h : W \longrightarrow M.$$

Then h_ε is smooth, and is the identity map on $X \subset N(X, M)$. Moreover, according to the decomposition of $T_{(x,0)}N(X, M)$ above, $(dh_\varepsilon)_{(x,0)}$ maps $T_{(x,0)}N(X, M)$ bijectively onto T_xM. So the theorem follows from GIFT. □