LECTURE 11: APPROXIMATIONS AND DEFORMATIONS

1. Whitney Approximation Theorems

¶ Approximation of functions.

We begin with the following application of P.O.U.: [In the special case when M is compact, this also follows from the Stone-Weierstrass approximation theorem.]

Theorem 1.1 (Whitney approximation theorem for continuous functions)**.**
Let M be a smooth manifold. Then for any continuous function $g : M \to \mathbb{R}$ and any positive continuous function $\delta : M \to \mathbb{R}_{>0}$, there exists a smooth function $f : M \to \mathbb{R}$ so that $|f(p) - g(p)| < \delta(p)$ holds for all $p \in M$.

We will prove a stronger version of this theorem. We need a definition:

Definition 1.2. We say a function $g : M \to \mathbb{R}$ is *smooth on* a subset $A \subset M$ if there exists an open set $U \supset A$ and a smooth function g_0 defined on U s.t. $g_0 = g$ on A.

As a consequence, any function g is smooth on any single point set $\{p\}$, although it may fail to be smooth at p. Now we state the relative version of Theorem 1.1:

Theorem 1.3 (Whitney approximation theorem for functions, relative version)**.**
Let M be a smooth manifold, and $A \subset M$ a closed subset. Then for any continuous function $g : M \to \mathbb{R}$ which is smooth on A and any positive continuous function $\delta : M \to \mathbb{R}_{>0}$, there exists a smooth function $f : M \to \mathbb{R}$ with $f = g$ on A, so that $|f(p) - g(p)| < \delta(p)$ holds for all $p \in M$.

Proof. [The idea of the proof: For each p one can find a tiny small open set U_p containing p so that g is "almost constant" on U_p. Then on U_p one can approximate g by the constant function $f(\cdot) \equiv g(p)$ (on U_p). Then "glue" all these constant functions together via a P.O.U. ρ_p subordinate to the open cover $\{U_p\}$.] By definition, there exists an open set $U \supset A$ and a smooth function g_0 defined on U so that $g_0 = g$ on A. Let $U_0 = \{p \in U : |g_0(p) - g(p)| < \delta(p)\}$. Then U_0 is open in M and $U_0 \supset A$.

Next we construct a (nice) open cover of $M \setminus A$. For any $q \in M \setminus A$, we let

$$U_q = \{p \in M \setminus A \ : \ |g(p) - g(q)| < \delta(p)\}.$$

Then $\{U_q \mid q \in M \setminus A\}$ is an open covering of $M \setminus A$. Since the topology on M is second countable, one can find countable many such $U_{q_i}, i = 1, 2, \cdots$, which cover $M \setminus A$.

Now let $\{\rho_0, \rho_i\}$ be P.O.U. subordinate to the open cover $\{U_0, U_{q_i} : i = 1, 2, \cdots\}$ of M, and define a smooth function on M via

$$f(p) = \rho_0(p) g_0(p) + \sum_{i \geq 1} \rho_i(p) g(q_i).$$

Since the summation is locally finite, f is smooth. Also by definition, $f = g_0 = g$ on A. Moreover, for any $q \in M$ one has

$$\begin{aligned}
|f(p) - g(p)| &= \left|\rho_0(p)g_0(p) + \sum_{i\geq 1}\rho_i(p)g(q_i) - \sum_{i\geq 0}\rho_i(p)g(p)\right| \\
&\leq \rho_0(p)|g_0(p) - g(p)| + \sum_{i\geq 1}\rho_i(p)|g(q_i) - g(p)| \\
&< \rho_0(p)\delta(p) + \sum_{i\geq 1}\rho_i(p)\delta(p) \\
&= \delta(p),
\end{aligned}$$

where in the last inequality, the fact $\rho_0(p)|g_0(p) - g(p)| < \rho_0(p)\delta(p)$ follows from the facts that if $p \in U_0$, then by definition $|g_0(p) - g(p)| < \delta(p)$, while if $p \notin U_0$, then $\rho_0(p)$=0; the fact $\rho_i(p)|g_i(q) - g(p)| < \rho_i(p)\delta(p)$ follows from a similar argument on whether p is in U_{q_i} or not (and note that at least one of the inequality holds). $\square$

By working on each component, it is obvious that the same conclusion holds for any $\mathbb{R}^K$-valued maps:

Theorem 1.4 (Whitney approximation theorem for $\mathbb{R}^K$-valued maps, relative version)**.** *Let M be a smooth manifold, and $A \subset M$ a closed subset. Then for any continuous map $g : M \to \mathbb{R}^K$ which is smooth on A and any positive continuous function $\delta : M \to \mathbb{R}_{>0}$, there exists a smooth map $f : M \to \mathbb{R}^K$ with $f = g$ on A, so that $|f(p) - g(p)| < \delta(p)$ holds for all $p \in M$.*

Remark. (1) By taking $A = \emptyset$ we see that Theorem 1.3 implies Theorem 1.1.

(2) Theorem 1.3 also implies a smooth version of Tietze extension theorem:

> [**Smooth Extension**] Any smooth function on a closed subset A can be extended to a smooth function on the whole manifold M.

Note that our definition "smooth on a closed subset" is quite strong. There is an extension theorem under much weaker assumption, also due to Whitney:

> **Theorem 1.5** (Whitney extension theorem)**.** *Let $A \subset \mathbb{R}^n$ be a closes subset, and $\{f_\alpha\}_{|\alpha|\leq m}$ is a collection of functions defined on A which are* compatible *in the following sense:*
>
> $$f_\alpha(x) = \sum_{|\beta|\leq m-|\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta + o(|x-y|^{m-|\alpha|}).$$
>
> *Then there exists $f \in C^m(\mathbb{R}^n)$ so that*
> (1) *For all $|\alpha| \leq m$, $\partial^\alpha f = f_\alpha$ on A*
> (2) *f is real-analytic on $A^c = \mathbb{R}^n \setminus A$.*

¶ Approximate continuous maps by smooth maps [in homotopy class].

Now let M, N be smooth manifolds and let $g : M \to N$ be a continuous map. A natural question to ask is: Can we "approximate" g by a smooth map $f : M \to N$? Of course there is one issue in proposing the above question:

What do we mean by "approximate $g \in C^0(M,N)$ by $f \in C^\infty(M,N)$"?

Usually there are two different meanings:

- Endow with N a metric d [e.g. by embedding N into Euclidean space so that it admits an induced metric], so that $C^0(M,N)$ is a metric space (endowed with the uniform metric d_u)
- f can be obtained by "deforming" the map g.

In what follows we will mainly work on the second meaning (however, we will see the use of the first meaning in the proof). We will not work directly on the first meaning, since the approximation depends heavily on the extra structure: the metric or the embedding. (For example, for any $\varepsilon > 0$, we can embed N^n into an ε-ball in $\mathbb{R}^{2n+1}$, so that any smooth map $f \in C^\infty(M,N)$ is a ε-approximation of any continuous map $g \in C^0(M,N)$, which is not what we want.)

Recall from topology that "continuous deformations" are equivalent to "homotopies" [For details, see my topology (H) notes. Note that manifolds are always locally compact Hausdorff]. Recall that two maps $f_0, f_1 \in C^0(X,Y)$ are *homotopic* if there exists $F \in C^0(X \times [0,1], Y)$ so that

$$F(\cdot,0) = f_0(\cdot) \quad \text{and} \quad F(\cdot,1) = f_1(\cdot).$$

Now we prove the Whitney Approximation Theorem for continuous maps, which claims that any continuous map between smooth manifolds can be continuously deformed to a smooth map:

Theorem 1.6 (Whitney Approximation Theorem for Continuous Maps)**.**
Given any continuous mapping $g \in C^0(M,N)$, one can find a smooth mapping $f \in C^\infty(M,N)$ which is homotopic to g. Moreover, if g is smooth on a closed subset $A \subset M$, then one can choose f so that $f = g$ on A.

Proof. We embed N into $\mathbb{R}^K$. By the ε-neighborhood theorem we proved last time, there is a continuous function $\varepsilon : N \to \mathbb{R}_{>0}$ so that each $y \in N^\varepsilon$ possesses a unique closest point $\pi_\varepsilon(y) \in N$.

Think of g as a continuous function from M to $\mathbb{R}^K$ and apply Theorem 1.4 to the positive continuous function $\overline{\varepsilon} = \varepsilon \circ g$, we get a smooth map $\tilde{f} : M \to \mathbb{R}^K$ which is ε-close to g, i.e.

$$|\tilde{f}(x) - g(x)| < \varepsilon(g(x)), \quad \forall x \in M.$$

So $\tilde{f}(x) \in B(g(x), \varepsilon(g(x))) \subset N^\varepsilon$. It follows that

$$(1-t)g(x) + t\tilde{f}(x) \in B(g(x), \varepsilon(g(x))) \subset N^\varepsilon, \quad \forall\, 0 \le t \le 1.$$

Now define $F : M \times [0,1] \to N$ by

$$F(x,t) = \pi_\varepsilon((1-t)g(x) + t\tilde{f}(x)).$$

Then F is a homotopy that connects the continuous map g to the smooth map

$$f = F(\cdot,1) = \pi_\varepsilon \circ \tilde{f} : M \to N.$$

Finally note that if g is smooth on a closed subset A, then the smooth function $\tilde{f}$ can be chosen so that $\tilde{f} = g$ on A. It follows that $f = g = F(\cdot, t)$ on A. (In other words, the homotopy connecting g to f can be chosen to be relative to A.) □

Remark. If M, N are real analytic manifolds, then one can approximate continuous maps by analytic maps.

As an immediate consequence, we prove

Corollary 1.7. *The homotopy group $\pi_k(S^n) \simeq \{0\}$ if $k < n$.*

Proof. Any continuous map $f : S^k \to S^n$ is homotopic to a smooth map $\tilde{f} : S^k \to S^n$. Since $k < n$, by Sard's theorem, $\tilde{f}(S^k)$ is of measure zero in S^n. In particular, $\tilde{f}$ is not surjective, and thus is null-homotopic (why?). □

2. Smooth deformation of smooth maps

¶ Smooth homotopy.

Since in this course, we are mainly interested in smooth objects (smooth manifolds, smooth submanifolds, smooth functions, smooth maps, smooth vector fields, smooth vector bundle, smooth differential form etc), we are interested in homotopies connecting two smooth maps via "smooth path", i.e.

Definition 2.1. We say $f_0, f_1 \in C^\infty(M, N)$ are *smoothly homotopic* if there exists $F \in C^\infty(M \times [0, 1], N)$ so that

$$F(\cdot, 0) = f_0 \quad \text{and} \quad F(\cdot, 1) = f_1.$$

Of course if f_0 and f_1 are smoothly homotopic, then they are homotopic. Conversely, we have

Theorem 2.2 (Homotopy $\equiv$ Smooth homotopy)**.**
Suppose $f_0, f_1 \in C^\infty(M, N)$ are homotopic, then they are smoothly homotopic.

Proof. Let $F : M \times [0, 1] \to N$ be a homotopy connecting f_0 and f_1. Continuously extend F to a mapping $\widetilde{F} : M \times \mathbb{R} \to N$ by defining

$$\widetilde{F}(x, t) = F(x, 0) \text{ if } t \leq 0, \quad \text{and} \quad \widetilde{F}(x, t) = F(x, 1) \text{ if } t \geq 1.$$

Then $\widetilde{F}$ is a continuous map from $M \times \mathbb{R}$ to N, and is smooth on closed subsets $M \times \{0\}$ and $M \times \{1\}$. (Note that by Definition 1.2, $\widetilde{F}$ is smooth on $M \times \{0\}$ means there is a smooth function $\widetilde{G}$ defined on $M \times (-\varepsilon, \varepsilon)$ so that $\widetilde{G}(x, 0) = \widetilde{F}(x, 0)$. We don't require $\widetilde{F}$ to be smooth in a neighborhood of $M \times \{0\}$.) So by Theorem 1.6, there exists a smooth map $\overline{F} : M \times \mathbb{R} \to N$ (that is homotopic to $\widetilde{F}$, which we don't need here), such that $\overline{F} = \widetilde{F}$ on $M \times \{0\}$ and $M \times \{1\}$, i.e.

$$\overline{F}(\cdot, 0) = f_0 \text{ and } \overline{F}(\cdot, 1) = f_1.$$

It follows that $\overline{F}$ is the desired smooth homotopy connecting f_0 and f_1. □

Recall that homotopy is an equivalence relation on the space of continuous maps from M to N. The equivalence classes are called homotopy classes of maps. Theorem 2.2 implies that smooth homotopy is an equivalence relation on the space of smooth maps from M to N. Moreover, combining Theorem 1.6 and Theorem 2.2, we immediately see that homotopy classes of continuous maps coincides with the smooth homotopy classes of smooth maps. But in general the smooth theory is easier to compute, since we have a power weapon: the differentiation.

¶ Stable properties.

In many applications, it is important that certain properties of maps will remain unchanged under a small deformation.

Definition 2.3. We say a property $\mathcal{P}$ concerning maps in $C^\infty(M,N)$ is a *stable property* if it is preserved under small deformation, namely, if $f \in C^\infty(M,N)$ satisfies $\mathcal{P}$ and F is a smooth homotopy with $F(x,0) = f$, then there exists $\varepsilon > 0$ so that for each $0 < t < \varepsilon$, the map $f_t(\cdot) = F(\cdot, t)$ satisfies the property $\mathcal{P}$.

Theorem 2.4. *Suppose M is compact. Then the following properties of maps in $C^\infty(M,N)$ are stable:*

(1) *immersion,*
(2) *submersion,*
(3) *embedding,*
(4) *local diffeomorphism,*
(5) *diffeomorphism.*

Proof. (1) Let $f : M \to N$ be an immersion. Then for any $p \in M$, there is a $m \times m$ sub-matrix of df_p which is non-singular. By continuity, there exists an open set $U_p \ni p$ and $[0, \varepsilon_p)$ so that the corresponding $m \times m$ sub-matrix of $(df_t)_q$ is non-singular for all $(q,t) \in U_p \times [0, \varepsilon_p)$. (Strictly speaking, here we are working inside a coordinate chart U around p so that $f(U)$ is contained in a coordinate chart around $f(p)$.) Now the set $\bigcup_p U_p \times [0, \varepsilon_p)$ is an open neighborhood of $M \times \{0\}$ inside $M \times [0,1]$. By the tube lemma in general topology, there exists $\varepsilon > 0$ so that

$$M \times [0, \varepsilon) \subset \bigcup_p U_p \times [0, \varepsilon_p).$$

As a consequence, for any $t < \varepsilon$, $(df_t)_p$ is injective for all $p \in M$, i.e. $f_t : M \to N$ is an immersion.

(2) The proof is almost the same as (1).

(3) Recall that a map from a compact manifold is an embedding if and only if it is an injective immersion. In view of (1), we only need to show: there exists $\varepsilon > 0$ so that for each $0 < t < \varepsilon$, the map f_t is injective. (The proof is just a modification of the proof to GIFT (compact case) that we did last time.) Suppose we can find $t_i \to 0$ and x_i, x_i' with $f_{t_i}(x_i) = f_{t_i}(y_i)$. By compactness, after passing to subsequences we may assume $x_i \to x_0$ and $x_i' \to x_0'$. Then $x_0 = x_0'$ since

$$f(x_0) = F(x_0, 0) = \lim_{i\to\infty} F(x_i, t_0) = \lim_{i\to\infty} F(x_i', t_0) = F(x_0', 0) = f(x_0').$$

This will give us the desired contradiction, since the $(m+1) \times (n+1)$ matrix

$$dG_{(x_0,0)} = \begin{pmatrix} df_{x_0} & * \\ 0 & 1 \end{pmatrix}$$

has rank $m+1$, which implies that G is an immersion at $(x_0, 0)$, and thus by the canonical immersion theorem, G is an immersion (and thus is injective) in a neighborhood of $(x_0, 0)$.

(4) This is a consequence of (1): A local diffeomorphism is merely an immersion between manifolds of the same dimension.

(5) Let $f : M \to N$ be a diffeomorphism. Then f maps each connected component of M diffeomorphically to a connected component of N, and F maps a connected component of M into the corresponding connected component of N. So without loss of generality, we may assume M is connected.

Since a diffeomorphism is automatically an embedding, a submersion and a local diffeomorphism. By (2), (3) and (4), there exists $\varepsilon > 0$ os that for each $0 < t < \varepsilon$, the map f_t is an embedding (and thus is injective), an submersion and is a local diffeomorphism. Also recall that in PSet 2-2-1, we showed that any submersion from a compact manifold to a connected manifold is surjective. So each f_t is a diffeomorphism for $0 < t < \varepsilon$. □

Remark. However, being "constant rank" is not a stable property. (why?)

Remark. The properties listed in Theorem 2.4 fails to be stable if M is non-compact. See PSet for a counterexample.