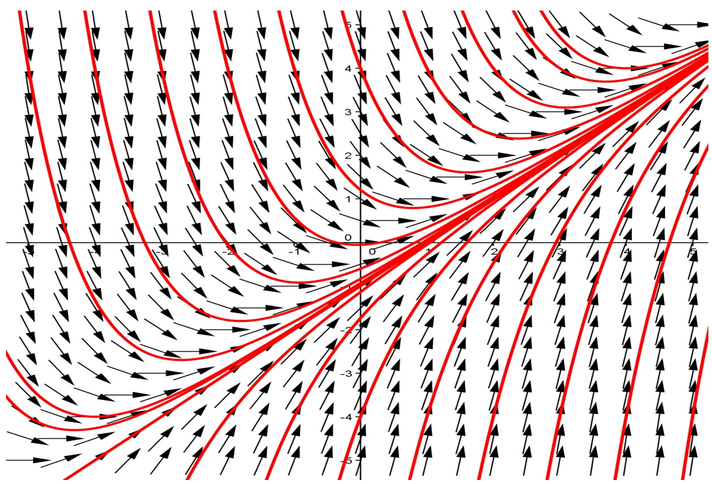
LECTURE 14: INTEGRAL CURVES OF SMOOTH VECTOR FIELDS

1. Integral Curves

¶ Integral Curves.

Suppose we have a smooth vector field defined on an Euclidian region. In calculus and in ODE, we learned the conception of integral curves of such a vector field: an integral curve is a parametric curve that represents a specific solution to the ordinary differential equation represented by the vector field. Geometrically, they are curves so that the given vector field is the tangent vector to the curves everywhere.

Here is an example of vector fields with many integral curves drawn:



The conception of integral curves above can be generalized to smooth manifolds easily. Recall that a *smooth curve* in a smooth manifold M is a smooth map $\gamma : I \to M$, where I is an interval in $\mathbb{R}$. For any $a \in I$, the *tangent vector* of γ at the point $\gamma(a)$ is

$$\dot{\gamma}(a) = \frac{d\gamma}{dt}(a) := d\gamma_a(\frac{d}{dt}),$$

where $\frac{d}{dt}$ is the standard coordinate tangent vector of $\mathbb{R}$.

Definition 1.1. Let $X \in \Gamma^\infty(TM)$ be a smooth vector field on M. A smooth curve $\gamma : I \to M$ is called an *integral curve* of X if for any $t \in I$,

$$\dot{\gamma}(t) = X_{\gamma(t)}.$$

Remark. By a *curve* we really mean a "parametrized curve". The parametrization is a part of the definition. Different parametrizations of the "same geometric picture" represent different curves. In general, a re-parametrization of an integral curve is no longer an integral curve.

¶ Examples of integral Curves.

However, it is easy to see that linear re-parametrizations produce integral curves:

Lemma 1.2. *If $\gamma : I \to M$ is an integral curve of a vector field X, then*

(1) Let $I_a = \{t \mid t + a \in I\}$, then

$$\gamma_a : I_a \to M, \quad \gamma_a(t) := \gamma(t + a)$$

is an integral curve of X.

(2) Let $I^a = \{t \mid at \in I\}$ $(a \neq 0)$, then

$$\gamma^a : I^a \to M, \quad \gamma^a(t) := \gamma(at)$$

is an integral curve for $X^a = aX$.

The proofs are simple and thus are omitted.

Example. Consider the coordinate vector field $X = \frac{\partial}{\partial x^1}$ on $\mathbb{R}^n$. Then the integral curves of X are the straight lines parallel to the x^1-axis, parametrized as

$$\gamma(t) = (c_1 + t, c_2, \cdots, c_n).$$

To check this, we note that for any smooth function f on $\mathbb{R}^n$,

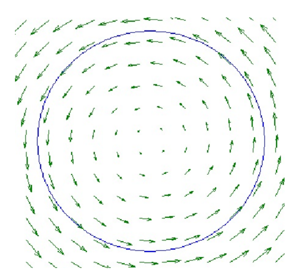
$$d\gamma(\frac{d}{dt})f = \frac{d}{dt}(f \circ \gamma) = \nabla f \cdot \frac{d\gamma}{dt} = \frac{\partial f}{\partial x^1}.$$

Remark. Note that although the curve

$$\tilde{\gamma}(t) = (c_1 + 2t, c_2, \cdots, c_n)$$

has the same picture (i.e. the same "horizontal line" passing the point $(c_1, \cdots, c_n)$) as γ, it is not an integral curve of X, but an integral curve of $2X$, since $\dot{\tilde{\gamma}}(t) = 2\frac{\partial}{\partial x^1}$.

Example. Consider the vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$. Then if $\gamma(t) = (x(t), y(t))$ is an integral curve of X, we must have, for any $f \in C^\infty(\mathbb{R}^2)$,



$$x'(t)\frac{\partial f}{\partial x} + y'(t)\frac{\partial f}{\partial y} = \nabla f \cdot \frac{d\gamma}{dt} = X_{\gamma(t)}f = x(t)\frac{\partial f}{\partial y} - y(t)\frac{\partial f}{\partial x},$$

which is equivalent to the system

$$x'(t) = -y(t), \quad y'(t) = x(t).$$

The solution to this system is

$$x(t) = a\cos t - b\sin t, \quad y(t) = a\sin t + b\cos t.$$

These are circles centered at the origin in the plane parametrized by the angle (with counterclockwise orientation).

¶ ODE in local charts: Existence, Uniqueness and Smoothness.

To study further properties of integral curves, we need to convert the equation $\dot\gamma(t) = X_{\gamma(t)}$ (which is an equation relating tangent vectors on manifolds) into ODEs on functions defined on Euclidian region. To do so we first notice the following nice local formula for a vector field, whose proof is left as an exercise:

Lemma 1.3. *Let X be a smooth vector field on M. Then in local chart (φ, U, V) we have $X = \sum X(x^i)\partial_i$, where $x^i : U \to \mathbb{R}$ is the i^{th} coordinate function defined by φ.*

So we let $\gamma : I \to M$ be an integral curve of X. To study the equation $\dot\gamma(t) = X_{\gamma(t)}$ at a given point $\gamma(t)$, WLOG we may assume $\gamma(t) \in U$, and (φ, U, V) is a coordinate chart. By using the local chart map φ, one can convert the point $\gamma(t) \in U$ to

$$\varphi(\gamma(t)) = (x^1(\gamma(t)), \cdots, x^n(\gamma(t))) \in \mathbb{R}^n.$$

If we denote $y^i = x^i \circ \gamma : I \to \mathbb{R}$, then we can convert the (vector!) equation defining integral curves into equations on these one-variable functions y^i's. More precisely, according to the previous lemma, we have

$$\dot\gamma(t) = d\gamma_t(\frac{d}{dt}) = \sum_i d\gamma_t(\frac{d}{dt})(x^i)\partial_i = \sum_i (x^i \circ \gamma)'(t)\partial_i = \sum_i (y^i)'(t)\partial_i.$$

So the integral curve equation $\dot\gamma(t) = X_{\gamma(t)}$ becomes

$$\sum_i (y^i)'(t)\partial_i = \sum_i X^i(\gamma(t))\partial_i = \sum_i X^i \circ \varphi^{-1}(y^1(t), \cdots, y^n(t))\partial_i.$$

for all $t \in I$. In conclusion, we convert the integral curve equation into the following system of ODEs on the one-variable functions y^i's:

$$(y^i)'(t) = X^i \circ \varphi^{-1}(y^1, \cdots, y^n), \qquad \forall t \in I,\ \forall 1 \le i \le n.$$

This is a system of first order ODEs on the (one-variable) functions $y^i = x^i \circ \gamma : \mathbb{R} \to \mathbb{R}$. Conversely, any solution to this system of ODEs defines an integral curve of the vector field X inside the open set U.

Recall:

Theorem 1.4 (The Fundamental Theorem for Systems of First Order ODEs)**.** *Suppose $V \subset \mathbb{R}^n$ is open, and $F = (F^1, \cdots, F^n) : V \to \mathbb{R}^n$ a smooth vector-valued function. Consider the initial value problem*

$$\begin{cases} \dot y^i(t) = F^i(y^1(t), \cdots, y^n(t)), & i = 1, \cdots, n \\ y^i(t_0) = c^i, & i = 1, \cdots, n \end{cases} \tag{1}$$

Then

- *(1)* **Existence** : *For arbitrary $t_0 \in \mathbb{R}$ and $c_0 \in V$, there exist an open interval $I_0 \ni t_0$ and an open set $V_0 \ni c_0$ so that for any $c = (c^1, \cdots, c^n) \in V_0$, the system (1) has a smooth solution $y_c(t) = (y^1(t), \cdots, y^n(t)) \in V$ for $t \in I_0$.*
- *(2)* **Uniqueness** : *If y_1 is a solution to the system (1) for $t \in I_0$ and y_2 is a solution to the system (1) for $t \in J_0$, then $y_1 = y_2$ for $t \in I_0 \cap J_0$.*
- *(3)* **Smoothness** : *The solution function $Y(c, t) := y_c(t)$ in part (1) is smooth on $(c, t) \in V_0 \times I_0$.*

We will refer to Lee's book, Appendix D (Page 663-671) for a proof. According to the fundamental theorem of systems of ODEs, we conclude

Theorem 1.5 (Local Existence, Uniqueness and Smoothness)**.** *Suppose X is a smooth vector field on M. Then for any point $p \in M$, there exists a neighborhood U_p of p, an $\varepsilon_p > 0$ and a smooth map*

$$\Gamma : (-\varepsilon_p, \varepsilon_p) \times U_p \to M$$

so that for any $q \in U$, the curve $\gamma_q : (-\varepsilon, \varepsilon) \to M$ defined by

$$\gamma_q(t) := \Gamma(t, q)$$

is an integral curve of X with $\gamma(0) = q$. Moreover, this integral curve is unique in the sense that if $\sigma : I \to M$ is another integral curve of X with $\sigma(0) = q$, then $\sigma(t) = \gamma_q(t)$ for $t \in I \cap (-\varepsilon_p, \varepsilon_p)$.

2. Complete vector fields

¶ Complete/Noncomplete vector fields.

As a consequence of the uniqueness, any integral curve has a *maximal defining interval*. We are interested in those vector fields whose maximal defining interval is $\mathbb{R}$.

Definition 2.1. A vector field X on M is *complete* if for any $p \in M$, there is an integral curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$.

Note every smooth vector field is complete.

Example. Consider the vector field $X = t^2 \frac{d}{dt}$ on $\mathbb{R}$. Let $\gamma(t) = (x(t))$ be its integral curve. Then according to the integral curve equation,

$$x'(t)\frac{d}{dt} = X_{\gamma(t)} = x(t)^2 \frac{d}{dt} \quad \Longrightarrow x'(t) = x(t)^2.$$

The solution to this ODE is with initial condition $x(0) = c$ is

$$x_c(t) = \frac{1}{-t + 1/c} \quad \text{for } c \neq 0$$

and

$$x_0(t) = 0 \quad \text{for } c = 0.$$

Note that the maximal interval of $x_c(t)$ is

$$I_c = (-\infty, 1/c) \quad \text{for } c > 0$$

and

$$I_c = (1/c, +\infty) \quad \text{for } c < 0.$$

Since the integral curves starting at any $c \neq 0$ is not defined for all $t \in \mathbb{R}$, we conclude that X is not complete.

¶ Compactly supported vector fields are complete.

We will use complete vector fields to construct global flows next time. We will end this lecture with a sufficient condition for a vector field to be complete. As in the case of functions, we can define the *support* of a vector field by

$$\mathrm{supp}(X) = \overline{\{p \in M \mid X(p) \neq 0\}}.$$

Our criteria is

Theorem 2.2. *If X is a compactly supported vector field on M, then it is complete.*

Proof. Let $C = \mathrm{supp}(X)$. First suppose $q \in M \setminus C$, i.e. $X_q = 0$. We define a "constant curve" γ_q on M by letting $\gamma_q(t) = q$ for all $t \in \mathbb{R}$, then we see

$$\dot{\gamma}_q(t) = 0 = X_q = X_{\gamma_q(t)}.$$

In other words, the constant curve γ_q (whose domain is $\mathbb{R}$) is the unique integral curve of X passing q.

Now suppose $p \in C$. (The idea: use compactness to find a uniform constant ε_0 so that any integral curve starting at a point in C is defined on an interval of length $2\varepsilon_0$. Then if we have an integral curve, we can always extend the domain by ε_0.) Since any integral curve starting at $q \in M \setminus C$ stays at q, we see that every integral curve starting at $p \in C$ stays in C. By Theorem 1.5, for any $q \in C$, there is an interval $I_q = (-\varepsilon_q, \varepsilon_q)$, a neighborhood U_q of q and a smooth map

$$\Gamma : I_q \times U_q \to C$$

such that for all $p \in U_q$,

$$\gamma_p(t) = \Gamma(t, p)$$

is an integral curve of X with $\gamma_p(0) = p$. Since $\cup_q U_q \supset C$, and C is compact, one can find a finite many points $q_1, \cdots q_N$ in C so that $\{U_{q_1}, \cdots, U_{q_N}\}$ cover C. Let $I = \cap_k I_{q_k} = (-\varepsilon_0, \varepsilon_0)$, then for any $q \in C$, there is an integral curve $\gamma_q : I \to C$. Now suppose the maximal defining interval for $p \in C$ is I_p. We need to prove $I_p = \mathbb{R}$. In fact, if $I_p \neq \mathbb{R}$, WLOG, we may assume that $\sup I_p = c < \infty$. Then starting from the point $q = \gamma_p(c - \frac{\varepsilon_0}{2})$, there is an integral curve

$$\gamma_q : (-\varepsilon_0, \varepsilon_0) \to M$$

of the vector field X. By uniqueness, $\gamma_q(t) = \gamma_p(t + c - \frac{\varepsilon_0}{2})$. It follows that the defining interval of γ_p extends to $c + \frac{\varepsilon_0}{2}$, which is a contradiction. □

In particular, if M itself is compact, then the set $\mathrm{Supp}(X)$, as a closed set in the compact manifold, is always compact. So we get

Corollary 2.3. *Any smooth vector field on a compact manifold is complete.*

¶ The flow generated by a complete vector field.

Now suppose M is a smooth manifold and X is a complete vector field on M. By definition, for any $p \in M$, there is a unique integral curve $\gamma_p : \mathbb{R} \to M$ such that $\gamma_p(0) = p$. From this one can, for any $t \in \mathbb{R}$, define a map

$$\phi_t : M \to M, \quad p \mapsto \gamma_p(t).$$

By definition, $\phi_t(p)$ is smooth in t for any fixed p. In fact, the family of maps $\{\phi_t \mid t \in \mathbb{R}\}$ satisfies the following very important *group law*:

Proposition 2.4. *For any $t, s \in \mathbb{R}$, we have $\phi_t \circ \phi_s = \phi_{t+s}$.*

Proof. Notice that for any $p \in M$ and any fixed $s \in \mathbb{R}$, the two curves

$$\gamma_1(t) = \phi_t \circ \phi_s(p) \quad \text{and} \quad \gamma_2(t) = \phi_{t+s}(p)$$

are both integral curves for X starting at the same point

$$\gamma_1(0) = \phi_s(p) = \gamma_2(0).$$

By uniqueness of integral curves, we have

$$\phi_t \circ \phi_s = \phi_{t+s}.$$

□

Since $\phi_0 = \mathrm{Id}$, we conclude that

Corollary 2.5. *$\phi_t : M \to M$ is bijective, and $\phi_t^{-1} = \phi_{-t}$.*

Next time we will prove that each ϕ_t, as a map from M to M, is smooth, and thus is a diffeomorphism. As a consequence, the map

$$t \mapsto \phi_t$$

is a group homomorphism from $\mathbb{R}$ to $\mathrm{Diff}(M)$. We will call the family $\{\phi_t \mid t \in \mathbb{R}\}$ a *one-parameter subgroup of diffeomorphisms*.

In fact, the result we are going to prove below is a bit stronger: if we let Φ be the collection of all ϕ_t's,

$$\Phi : \mathbb{R} \times M \to M, \quad (t, p) \mapsto \phi_t(p).$$

We will show that the map Φ is smooth as a map on joint variables (t, p).