LECTURE 16: DISTRIBUTIONS AND FOLIATIONS

1. Distributions

¶ Distributions.

Suppose M is an n-dimensional smooth manifold. We have seen that any smooth vector field X on M can be integrated locally near any point p to an integral curve γ_p. Moreover,

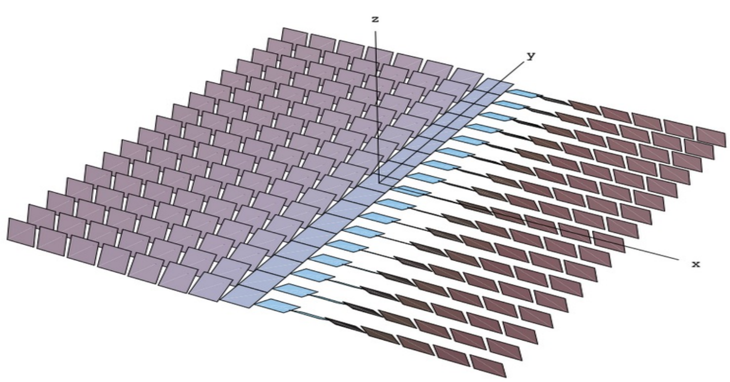
- If $X_p = 0$, then γ_p is the constant curve $\gamma_p(t) \equiv p$. (0-dimensional points)
- If $X_p \neq 0$, then γ_p is a 1-dimensional curve passing p. (1-dimensional curves)

In what follows we will develop a higher dimensional analogue to the "vector field $\leftrightsquigarrow$ integral curve" correspondence. We first generalize the conception of vector field.

Definition 1.1. Let M be a smooth manifold.

(1) A k-dimensional *distribution* $\mathcal{V}$ on M is a map which assigns to every point $p \in M$ a k-dimensional vector subspace $\mathcal{V}_p$ of T_pM.
(2) A distribution $\mathcal{V}$ is called *smooth* if for every $p \in M$, there is a neighborhood U of p and k smooth vector fields $X_1, \cdots, X_k$ on U such that for every $q \in U$, $\{X_1(q), \cdots, X_k(q)\}$ form a basis of $\mathcal{V}_q$. (In particular, $X_i(q) \neq 0$ for all $1 \leq i \leq k$.)
(3) We say a vector field X *belongs to* a distribution $\mathcal{V}$ if $X_p \in \mathcal{V}_p$ for all $p \in M$.

In what follows, all distributions will be smooth.



We also generalize the conception of integral curves:

Definition 1.2. Suppose $\mathcal{V}$ is a k-dimensional smooth distribution on M.

(1) An immersed submanifold $\iota : N \hookrightarrow M$ is called an *integral manifold* for $\mathcal{V}$ if for every $p \in N$, the image of $d\iota_N : T_pN \to T_pM$ is $\mathcal{V}_p$.
(2) We say the distribution $\mathcal{V}$ is *integrable* if through each point of M there exists an integral manifold of $\mathcal{V}$.

¶ Examples of integrable and non-integrable distributions.

Example. The vector fields $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}$ span a k-dimensional distribution $\mathcal{V}$ in $\mathbb{R}^n$. The integral manifolds of $\mathcal{V}$ are "hyperplanes" defined by the system of equations

$$x^i = c^i \quad (k+1 \le i \le n).$$

We will see below (Theorem 2.2, the local Frobenius theorem) that locally any integrable distribution can be written as this form.

Example. An integral manifold need not to be an embedded submanifold of M. For example, consider $M = S^1 \times S^1 \subset \mathbb{R}^2_x \times \mathbb{R}^2_y$. Fix any irrational number a, the integral manifolds of the non-vanishing vector field

$$X^a = (x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}) + a(y^2 \frac{\partial}{\partial y^1} - y^1 \frac{\partial}{\partial y^2})$$

are "dense curves" in M. (However, they are immersed submanifolds.)

Example. Any non-vanishing vector field X is a 1-dimensional distribution. It is always integrable: the image of any integral curve of X is an integral manifold.

It turns out that a higher dimensional distribution may fail to be integrable.

Example. Consider the smooth distribution $\mathcal{V}$ on $\mathbb{R}^3$ spanned by two vector fields

$$X_1 = \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^3}, \qquad X_2 = \frac{\partial}{\partial x^2}.$$

I claim that there is no integral manifold through the origin. In fact, if $\mathcal{V}$ is integrable, then the integrable manifold N of $\mathcal{V}$ containing the origin must also contain the integrable curve of X_1 passing the origin (for a reason, see Corollary 2.3 below), which is a piece of the x^1-axis, i.e.

$$N \supset \{(t, 0, 0) \mid |t| < \varepsilon\}.$$

Similarly N must also contain the integral curves of the vector field X_2 passing all these points $(t, 0, 0)$. It follows that for each $|t| < \varepsilon$, N contains a small piece of line segment parallel to the x^2-axis, i.e.

$$N \supset \{(t, s, 0) \mid |t| < \varepsilon, |s| < \delta_t.\}$$

In other words, N contains a piece of the x^1-x^2 plane that contains the origin. This is a contradiction, because the vector $\frac{\partial}{\partial x^1}$ is a tangent vectors of this piece of plane but is not in $\mathcal{V}_p$ for any $p \neq (t, 0, 0)$.

¶ Frobenius condition and involutive distributions.

Since not all distributions are integrable, we are interested in the conditions to make a distribution integrable. A necessary condition is easy to find. Intuitively, if two vector fields of M are tangent to a submanifold N of M, then their Lie bracket should also tangent to N. Motivated by this fact, we have

Theorem 1.3. *If a distribution $\mathcal{V}$ is integrable, then for any two vector fields X and Y belonging to $\mathcal{V}$, their Lie bracket $[X, Y]$ belongs to $\mathcal{V}$ also.*

Proof. Fix any $p \in M$, suppose $\iota : N \hookrightarrow M$ is an integrable manifold of $\mathcal{V}$ passing p. Since N is an immersed submanifold of M, one can "shrink" N so that $\iota(N)$ is in fact an embedded submanifold of M. Now suppose X, Y are vector fields belonging to $\mathcal{V}$, then the restrictions $X|_N, Y|_N$ of X, Y to N are vector fields that are tangent to the submanifold N. More precisely, there exists $\widetilde{X}, \widetilde{Y} \in \Gamma^\infty(TN)$ such that

$$X_p = d\iota_p(\widetilde{X}_p) \text{ and } Y_p = d\iota_p(\widetilde{Y}_p), \qquad \forall p \in N = \iota(N).$$

It follows from today's PSet (PSet 5-1-3-a-ii)) that for any $p \in N$,

$$[X, Y]_p^M = d\iota_p([\widetilde{X}, \widetilde{Y}]_p^N) \in \mathcal{V}_p.$$

So $[X, Y]$ belongs to $\mathcal{V}$ also. □

Definition 1.4. A distribution $\mathcal{V}$ is *involutive* if it satisfies the following *Frobenius condition*:

$$\boxed{\textbf{Frobenius condition}\text{: If } X, Y \in \Gamma^\infty(TM) \text{ belong to } \mathcal{V}\text{, so is } [X, Y].}$$

Example. Any 1 dimensional distribution is involutive since

$$[fX, gX] = (fX(g) - gX(f))X$$

is a multiple of X.

Note that by definition, to check a distribution $\mathcal{V}$ is involutive, one need to check $[X, Y] \in \mathcal{V}$ for all $X, Y \in \mathcal{V}$. It turns out that it is enough to check this for a set of local smooth basis of $\mathcal{V}$:

Lemma 1.5. *Let $\mathcal{V}$ be a k dimensional distribution on M. Suppose for each $p \in M$, there exist a neighborhood U of p and k pointwise linearly independent smooth vector fields $X_1, \cdots, X_k$ on U so that $[X_i, X_j]$ belong to $\mathcal{V}$ for all $1 \le i, j \le k$. Then $\mathcal{V}$ is involutive.*

Proof. Left as an exercise. □

Example. The distribution spanned by $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}$ in $\mathbb{R}^n$ is involutive, since

$$[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0, \qquad \forall 1 \le i, j \le k.$$

Example. The distribution $\mathcal{V}$ spanned by

$$X_1 = \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^3}, \quad X_2 = \frac{\partial}{\partial x^2}$$

is not involutive, since

$$[X_1, X_2] = -\frac{\partial}{\partial x^3}$$

is not in $\mathcal{V}$.

¶ Frobenius Theorem.

Theorem 1.3 tells us that any integrable distribution is involutive. It turns out that the converse is also true, namely, any involutive distribution is integrable:

Theorem 1.6 (Global Frobenius Theorem)**.** *Let $\mathcal{V}$ be an involutive k-dimensional distribution. Then through every point $p \in M$, there is a unique maximal connected integral manifold of $\mathcal{V}$ (in particular, $\mathcal{V}$ is integrable).*

Example. Let $f : M \to N$ be a submersion. For any $p \in M$, let $\mathcal{V}_p = \ker(df_p)$. Then

- $\mathcal{V}$ is a distribution since $\dim \mathcal{V}_p = \dim M - \dim N$ is constant for all $p \in M$.
- $\mathcal{V}$ is involutive: if X, Y are vector fields belonging to $\mathcal{V}$, then $df_p(X_p) = df_p(Y_p) = 0$ for all p. It follows that for any $g \in C^\infty(N)$, $X(g \circ f)(p) = df_p(X_p)(g) = 0$ for all p, i.e. $X(g \circ f) = 0$. Similarly $Y(g \circ f) = 0$. It follow that
$$df_p([X, Y]_p)(g) = [X, Y]_p(g \circ f) = X_p(Y(g \circ f)) - Y_p(X(g \circ f)) = 0.$$
- $\mathcal{V}$ is integrable: the integrable manifold passing $p \in M$ is the submanifold $f^{-1}(f(p))$.

Example. Consider the distribution $\mathcal{V}$ on $\mathbb{R}^3$ spanned by

$$X_1 = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}, \qquad X_2 = \frac{\partial}{\partial x^3}$$

on $M = \mathbb{R}^3 - \{x^1 = x^2 = 0\}$. Since $[X_1, X_2] = 0$, $\mathcal{V}$ is involutive. What are its integral manifolds? Well, let's first compute the integral curves of X_1 and X_2. Through any point (x^1, x^2, x^3), the integral curves of X_1 are circles in the x^3-plane with origin the center, and the integral curves of X_2 are the lines that are parallel to the x^3-axis. Note that the integral manifold passing (x^1, x^2, x^3) of the distribution should contain all points of the form $\varphi_t^{X_1}(\varphi_s^{X_2}(x^1, x^2, x^3))$ for all t, s. In our case, these are the cylinders centered at the x^3-axis.

¶ Foliation.

As a consequence, given any k-dimensional involutive distribution on M, one can "decompose" M into a disjoint union of k-dimensional connected immersed submanifolds. Such a decomposition is called a *foliation structure* on M.

Definition 1.7. A k-dimensional *foliation* $\mathcal{F}$ of an m-dimensional manifold M is a decomposition of M into a union of disjoint connected immersed submanifolds $\{L_\alpha\}_{\alpha \in A}$, called the *leaves of the foliation*, with the following property:

> Every point in M has a neighborhood U and a system of local coordinates $x = (x_1, \cdots, x_m)$ on U such that for each leaf L_α, the connected components of $U \cap L_\alpha$ are described by the equations $x_{k+1} = c_{k+1}, \cdots, x_m = c_m$.

A foliation is called *regular* if each leaf is an embedded submanifold.

Using the language of foliation, we can rewrite the Frobenius theorem as: Let $\mathcal{V}$ be an involutive k-dimensional distribution. Then the collection of all maximal connected integral manifold of $\mathcal{V}$ form a k-dimensional foliation $\mathcal{F}$ of M. Conversely, given any k-dimensional foliation $\mathcal{F}$, by definition the collection of all tangent spaces of all leaves form an integrable k-dimensional distribution.

2. The proof of Frobenius theorem

¶ Flatten one vector field locally.

We first prove the following useful lemma:

Lemma 2.1. *Let X be any smooth vector field on M with $X_p \neq 0$. Then there is a local chart (φ, U, V) near p so that $X = \partial_1$ on U.*

Proof. Choose a local chart $(\widetilde{U}, y^1, \cdots, y^n)$ around p such that $X_p = \frac{\partial}{\partial y^1}\Big|_p$. Denote $X = \sum_{i=1}^n \xi_i \frac{\partial}{\partial y^i}$ on $\widetilde{U}$, where ξ_i are smooth functions on $\widetilde{U}$. Shrinking $\widetilde{U}$ if necessary, we may assume $\xi_1 \neq 0$ on $\widetilde{U}$. Consider the system of ODEs

$$\frac{dy^i}{dy^1} = \frac{\xi_i(y^1; y^2, \cdots, y^n)}{\xi_1(y^1; y^2, \cdots, y^n)}, \quad 2 \leq i \leq n.$$

By basic theory of ODE, locally for any given initial data $(z^2, \cdots, z^n)$, with $|z| < \varepsilon$, the system above has a unique solution

$$y^i = y^i(y^1; z^2, \cdots, z^n), \quad |y^1| < \varepsilon$$

with initial condition

$$y^i(0; z^2, \cdots, z^n) = z^k, \quad 2 \leq i \leq n$$

and the functions y^i depends smoothly on y^1 and on z^j's. Consider

$$\begin{aligned} &y^1 = z^1, \\ &y^i = y^i(z^1; z^2, \cdots, z^n), \quad 2 \leq i \leq n. \end{aligned}$$

Since the Jacobian

$$\frac{\partial(y^1, \cdots, y^n)}{\partial(z^1, \cdots, z^n)}\Big|_{z^1=0} = 1,$$

we can make the change of variables from $(y^1, \cdots, y^n)$ to $(z^1, \cdots, z^n)$, i.e. there exists a neighborhood $U \subset \widetilde{U}$ of p, with $(z^1, \cdots, z^n)$ as local coordinate functions. We have in this new chart

$$X = \sum \xi_i \frac{\partial}{\partial y^i} = \xi_1 \sum \frac{\partial y_i}{\partial z^1}\frac{\partial}{\partial y^i} = \xi_1 \frac{\partial}{\partial z^1}.$$

Finally if we let $x^1(z^1, \cdots, z^n) = \int_0^{z_1} \frac{dt}{\xi_1(t, z^2, \cdots, z^n)}$ and $x_j = z_j$ for $j \geq 2$, then $\{x^1, \cdots, x^n\}$ are local coordinate functions on U such that $X = \frac{\partial}{\partial x^1}$ on U. □

Remark. More generally, if $[X_i, X_j] = 0$ and if X_i's are linearly independent, then there exists a coordinate chart so that $X_i = \partial_i$. The proof is left as an exercise. Note that the condition $[X_i, X_j] = 0$ is necessary since in any local charts, we have $[\partial_i, \partial_j] = 0$.

¶ Local Frobenius Theorem: "Flatten" an involutive distribution.

Before proving the global Frobenius theorem, we first prove the following local version: locally any involutive distribution is "flat", and thus is integrable.

Theorem 2.2 (Local Frobenius Theorem)**.** *Let $\mathcal{V}$ be an involutive k-dimensional distribution. Then for every $p \in M$, there exists a coordinate patch $(U, x^1, \cdots, x^n)$ centered at p such that for all $q \in U$, $\mathcal{V}_q = \mathrm{span}\{\partial_1(q), \cdots, \partial_k(q)\}$.*

Proof. By Lemma 2.1, the theorem holds for $k = 1$. Now suppose the theorem holds for $k-1$ dimensional involutive distributions. Let $\mathcal{V}$ be an k dimensional distribution spanned locally by $X_1, X_2, \cdots, X_k$ near p. Suppose $\mathcal{V}$ is involutive, i.e.

$$[X_i, X_j] \equiv 0 \mod (X_1, \cdots, X_k), \quad 1 \le i, j \le k.$$

According to Lemma 2.1, there exits a local chart $(U; y^1, \cdots, y^n)$ near p such that $X_k = \partial_{y^k}$. For $1 \le i \le k-1$ let

$$X_i' = X_i - X_i(y^k)X_k,$$

then $X_i'(y^k) = 0$ for $1 \le i \le k-1$, and $X_k(y^k) = 1$. Note that the vector fields $X_1', \cdots, X_{k-1}', X_k$ still span $\mathcal{V}$. Moreover, if we denote

$$[X_i', X_j'] = a_{ij}X_k \mod (X_1', \cdots, X_{k-1}'), \quad 1 \le i, j \le k-1,$$

then after applying both sides to the function y^k, we see $a_{ij} = 0$ for all $1 \le i, j \le k-1$. According to Lemma 1.5, the $k-1$ dimensional distribution

$$\mathcal{V}' = \mathrm{span}\{X_1', \cdots, X_{k-1}'\}$$

is involutive. By induction hypothesis, there is a local chart $(U, z^1, \cdots, z^n)$ near p such that $\mathcal{V}'$ is spanned by $\{\partial_{z^1}, \cdots, \partial_{z^{k-1}}\}$. Since each $\partial_{z^i} (1 \le i \le j)$ is a linear combination of X_j' for $1 \le j \le k-1$, we conclude $\partial_{z^i}(y^k) = 0$.

Now denote

$$[\partial_{z^i}, X_k] = b_i X_k \mod (\partial_{z^1}, \cdots, \partial_{z^{m-1}}).$$

Apply both sides to the function y^k, we see $b_i = 0$ for all i. So we can write

$$[\partial_{z^i}, X_k] = \sum_{j=1}^{k-1} C_j^i \partial_{z^j}.$$

Suppose $X_k = \sum_{j=1}^n \xi_j \partial_{z^j}$. Insert this into the previous formula, we see

$$\partial_{z^i}\xi_j = 0, \quad 1 \le i \le k-1, k \le j \le n.$$

In other words, for $j \ge k$, $\xi_j = \xi_j(z^k, \cdots, z^n)$. Let

$$X_k' = \sum_{j=k}^n \xi_j \partial_{z^j}.$$

Then $\{\partial_{z^1}, \cdots, \partial_{z^{k-1}}, X_k'\}$ still span $\mathcal{V}$. Finally according to Lemma 2.1 again, there is a local coordinate change from $(z^1, \cdots, z^k, \cdots, z^n)$ to $(x^1, \cdots, x^k, \cdots, x^n)$ with $x^i = z^i$ for $1 \le i \le k-1$, such that $X_k' = \partial_{x^k}$. This completes the proof. $\square$

As a consequence, we get the following result that we used in previous examples:

Corollary 2.3. *Let X be a smooth vector field belongs to $\mathcal{V}$. Then for any $p \in M$, the integral curve of X passing the point p lies in the integral manifold of $\mathcal{V}$ passing p.*

¶ Sketch of proof of the Global Frobenius theorem:

Sketch of proof of the Global Frobenius theorem: For any $p \in M$, let

$$N_p = \{q \in M \mid \exists \text{ a piecewise smooth integral curve of } \mathcal{V} \text{ jointing } p \text{ to } q\}^1.$$

We claim that N_p is the maximal connected integral manifold of $\mathcal{V}$ containing p.

The manifold structure is defined as follows: for any $q \in N_p$, there is a coordinate patch (ϕ, U, V) (with $\phi(w) = (x^1(w), \cdots, x^n(w))$) centered at q such that $\mathcal{V} = \mathrm{span}\{\partial_1, \cdots, \partial_k\}$ in U. For each small ε, let

$$W_\varepsilon = \{w \in U \mid (x^1)^2(w) + \cdots + (x^k)^2(w) < \varepsilon, x^{k+1}(w) = \cdots = x^n(w) = 0\}.$$

Then any point $w \in W_\varepsilon$ can be joint to q by the integral curve

$$\gamma(t) = \phi^{-1}(tx_1(w), \cdots, tx_k(w), 0, \cdots, 0).$$

So $W_\varepsilon \subset N_p$. Let

$$\varphi : W_\varepsilon \to B^k(\varepsilon) \subset \mathbb{R}^k, \quad w \mapsto (x_1(w), \cdots, x_k(w)).$$

Now we define the topology on N_p by giving it the weakest topology such that all these φ's are homeomorphisms. The atlas on N_p is defined to be the set of charts $(\varphi, W, B^k(\varepsilon))$. One can check that N_p is a manifold with this given atlas. For more details, c.f. Warner, pg.48-49. □

[1]We say γ is a piecewise smooth integral curve of $\mathcal{V}$ if γ is of the form "γ_1 connected to γ_2 connected $\cdots$ connected to γ_m", where each γ_i is a smooth integral curve of a smooth vector field in $\mathcal{V}$.