LECTURE 18: THE EXPONENTIAL MAP

1. Lie Homomorphisms

¶ Lie group/Lie algebra homomorphisms.

It is natural to study morphisms between Lie groups and between Lie algebras. As usual, they are maps between corresponding objects that preserves corresponding definition properties.

Definition 1.1. Let G, H be Lie groups.

(1) A map $\phi : G \to H$ is called a *Lie group homomorphism* if it is smooth and is a group homomorphism, i.e.

$$\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2), \qquad \forall g_1, g_2 \in G.$$

(2) A Lie group homomorphism $\phi : G \to H$ is called an *Lie group isomorphism* if it is invertible and the inverse $\phi^{-1} : H \to G$ is also a Lie group homomorphism.

Note that by definition, if two Lie groups are isomorphic, then they are diffeomorphic as manifolds, and are isomorphic as groups.

Example. For any Lie group G and any $a \in G$, the conjugation map

$$c(a) = L_a \circ R_{a^{-1}} : G \to G, \qquad g \mapsto aga^{-1}$$

is a Lie group isomorphism.

Remark. In PSet 1-1-1 we have seen that for any connected topological group G, there is an open neighborhood U of $e \in G$ such that $G = \cup_{n=1}^{\infty} U^n$. As a consequence, any Lie group homomorphism $\phi : G \to H$ is determined by its restriction to a neighborhood of $e \in G$, provided G is connected.

Similarly one can define Lie algebra homomorphisms to be

Definition 1.2. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras.

(1) A linear map $L : \mathfrak{g} \to \mathfrak{h}$ is called a *Lie algebra homomorphism* if

$$L([X_1, X_2]) = [L(X_1), L(X_2)], \qquad \forall X_1, X_2 \in \mathfrak{g}.$$

(2) A Lie algebra homomorphism $L : \mathfrak{g} \to \mathfrak{h}$ is called an *Lie algebra isomorphism* if it is invertible.

Note that if a Lie algebra homomorphism $L : \mathfrak{g} \to \mathfrak{h}$ is invertible, the inverse $L^{-1} : \mathfrak{h} \to \mathfrak{g}$ is automatically a Lie algebra homomorphism. So again we see the phenomena that "linear objects are much easier to handle".

Example. For any $X \in \mathrm{GL}(n, \mathbb{R})$, the adjoint map

$$\mathrm{Ad}_X : \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R}), \qquad A \mapsto XAX^{-1}$$

is a Lie algebra isomorphism. (Check this. What is $(\mathrm{Ad}_X)^{-1}$?)

¶ From Lie group homomorphisms to Lie algebra homomorphisms.

Now suppose $\phi : G \to H$ is a Lie group homomorphism, then its differential at e gives a linear map $d\phi_e : T_eG \to T_eH$. Under the identification

$$T_eG \simeq \mathfrak{g} \qquad \text{and} \qquad T_eH \simeq \mathfrak{h},$$

we get an induced map, which we will denote by $d\phi$, from $\mathfrak{g}$ to $\mathfrak{h}$:

$$d\phi : \mathfrak{g} \to \mathfrak{h}.$$

In other words, when viewed as left invariant vector fields, for any $X \in \mathfrak{g}$, the image $d\phi(X)$ is the left invariant vector field on H whose value at $e \in H$ is $d\phi_e(X_e)$, i.e.

$$(d\phi(X))_h = dL_h(d\phi_e(X_e)).$$

Example. Start with the conjugation map $c(X) : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ on $\mathrm{GL}(n, \mathbb{R})$. Taking differential at the identity matrix I_n, we get

$$(dc(X))_{I_n}(A) = \left.\frac{d}{dt}\right|_{t=0} c(X)(I + tA) == \left.\frac{d}{dt}\right|_{t=0} X(I + tA)X^{-1} = XAX^{-1}.$$

In other words, the induce map is the Lie algebra homomorphism

$$dc(X) = \mathrm{Ad}_X : \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R}).$$

In what follows we show that the induced map $d\phi$ is always a Lie algebra homomorphism. We need

Lemma 1.3. *Any $X \in \mathfrak{g}$ is ϕ-related to $d\phi(X) \in \mathfrak{h}$.*

Proof. Take any $X \in \mathfrak{g}$. Write $h = \phi(g)$. Since ϕ is a group homomorphism, we have

$$\phi \circ L_g = L_h \circ \phi.$$

It follows

$$d\phi_g(X_g) = d\phi_g \circ (dL_g)_e(X_e) = dL_h \circ d\phi_e(X_e) = (d\phi(X))_h.$$

This completes the proof. □

As a consequence, we can prove

Theorem 1.4. *If $\phi : G \to H$ is a Lie group homomorphism, then the induced map $d\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

Proof. Let $X, Y \in \mathfrak{g}$. According to the lemma above,

- X is ϕ-related to $d\phi(X)$, Y is ϕ-related to $d\phi(Y)$.
  $\rightsquigarrow$ So by PSet 5-1-3, $[X, Y]$ is ϕ-related to $[d\phi(X), d\phi(Y)]$.
- $[X, Y]$ is ϕ-related to $d\phi([X, Y])$.

It follows that

$$[d\phi(X), d\phi(Y)]_e = d\phi_e([X,Y]_e) = (d\phi([X,Y]))_e.$$

Since both $d\phi([X,Y])$ and $[d\phi(X), d\phi(Y)]$ are left-invariant vector fields on H, we conclude $d\phi([X,Y]) = [d\phi(X), d\phi(Y)]$. □

Example. Last time we have seen that $\mathrm{GL}(n,\mathbb{R})$ (with matrix multiplication) is a Lie group, and the corresponding Lie algebra is $\mathfrak{gl}(n,\mathbb{R})$ (with matrix commutator). It is also easy to see that $\mathbb{R}^*$ (with real number multiplication) is a Lie group $(= \mathrm{GL}(1,\mathbb{R}))$, and the corresponding Lie algebra is the trivial Lie algebra $\mathbb{R}$ (with real number addition) $(= \mathfrak{gl}(1,\mathbb{R}))$. It is easy to see that the determinant $\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^*$ is a Lie group homomorphism, since

$$\det(XY) = \det X \det Y, \qquad \forall X, Y \in \mathrm{GL}(n,\mathbb{R}).$$

According to PSet 2-2-4, we have

$$d\det{}_X(A) = (\det X)\mathrm{tr}(X^{-1}A), \qquad \forall X \in \mathrm{GL}(n,\mathbb{R}), A \in \mathfrak{gl}(n,\mathbb{R}).$$

By taking $X = \mathrm{I}_n$, we conclude that the induced Lie algebra homomorphism for det is

$$d\det = \mathrm{tr} : \mathfrak{gl}(n,\mathbb{R}) \to \mathbb{R}, \quad A \mapsto \mathrm{tr}(A).$$

Since $\mathbb{R}$ is the trivial Lie algebra, we thus get a conceptional proof of the fact

$$\mathrm{tr}(AB) = \mathrm{tr}(BA), \quad \forall A, B \in \mathfrak{gl}(n,\mathbb{R}).$$

Remark. In summary, we have two categories, the category $\mathcal{LIEGROUP}$

- Objects are Lie groups,
- Morphisms are Lie group homomorphisms

and the category $\mathcal{LIEALGEBRA}$

- Objects are Lie algebras,
- Morphisms are Lie algebra homomorphisms.

Moreover, we have a functor $\mathcal{LIE}$ that

- associates to each Lie group G its Lie algebra $\mathfrak{g}$,
- associates to Lie group homomorphism $\phi : G \to H$ the induced Lie algebra homomorphism $d\phi : \mathfrak{g} \to \mathfrak{h}$.[1]

Of course it is easy to find different Lie groups whose Lie algebras are the same. However, it turns out that the functor $\mathcal{LIE}$ is "invertible" when we are restricted to the subcategory of $\mathcal{LIEGROUP}$ whose objects are simply connected Lie groups: According to Lie's third theorem, any finitely dimensional Lie algebra is the Lie algebra of some simply connected Lie group; moreover, if G is simply connected, then any Lie algebra homomorphism $L : \mathfrak{g} \to \mathfrak{h}$ can be "lifted" to a Lie group homomorphism $\phi : G \to H$ so that $L = d\phi$.

[1]One need to check $d(\mathrm{Id}_G) = \mathrm{Id}_{\mathfrak{g}}$ and $d(\phi_1 \circ \phi_2) = d\phi_1 \circ d\phi_2$, both follows from definitions easily.

2. The Exponential Map

¶ The exponential map.

Let G be any Lie group, $\mathfrak{g}$ its Lie algebra. So by definition, any $X \in \mathfrak{g}$ is a left-invariant vector field. In general G need not be compact, and thus X need not be compactly-supported. However, thanks to the left translation maps, we have a "universal way" to control the vectors at different points of a left invariant vector field. As a result, one can prove(left as an exercise)

Lemma 2.1. *Any left invariant vector field $X \in \mathfrak{g}$ on a Lie group G is complete.*

Let $\phi_t^X : G \to G$ be the flow generated by $X \in \mathfrak{g}$. The completeness guarantees that ϕ_t^X is well-defined for all t. In particular, it is defined for $t = 1$.

Definition 2.2. The *exponential map*[2] of G is the map

$$\exp : \mathfrak{g} \to G, \quad X \mapsto \phi_1^X(e).$$

Remark. It is not hard to see $\phi_{ts}^X = \phi_s^{tX}$. So

$$\exp(tX) = \phi_1^{tX}(e) = \phi_t^X(e).$$

Moreover, one can easily show that $\{\exp(tX) | t \in \mathbb{R}\}$ is a *one-parameter subgroup of G*:

$$\exp(tX) \cdot \exp(sX) = \exp((t+s)X).$$

Note that in general

$$\exp(tX)\exp(tY) \neq \exp(t(X+Y)).$$

Example. For $G = \mathbb{R}^*$, we can identify $T_1G = \mathbb{R}$. For any $x \in \mathbb{R} = T_1G$, the left invariant vector field corresponding to $x = x\frac{d}{dt} \in T_1G$, when evaluated at $a \in G$, is

$$X_a = ax\frac{d}{dt}.$$

By solving corresponding ODE, one can find the integral curve of X starting at $e = 1$,

$$\gamma_e^X(t) = e^{tx}. \qquad (\dot{\gamma}_e^X(t) = xe^{tx}\frac{d}{dt} = X_{\gamma_e^X(t)})$$

So we get

$$\exp(x) = \phi_1^X(e) = \gamma_e^X(1) = e^x.$$

Example. Similarly one can show (exercise)

(1) for $G = (S^1, \cdot)$,

$$\exp : i\mathbb{R} = T_eS^1 \to S^1, \quad \exp(ix) = e^{ix},$$

(2) for $G = (\mathbb{R}^n, +)$,

$$\exp : \mathbb{R}^n = T_0\mathbb{R}^n \to \mathbb{R}^n, \quad \exp(x) = x,$$

[2]There is a conception of exponential map in Riemannian geometry. It turns out that if G is a compact Lie group endowed with the bi-invariant metric, then the Riemannian geometry exponential map coincide with the Lie theory exponential map.

(3) for $G = \mathrm{GL}(n, \mathbb{R})$,

$$\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}), \quad \exp(A) = e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots .$$

¶ The differential of the exponential map.

One of the most useful properties for the exponential map is

Lemma 2.3. *The exponential map* $\exp : \mathfrak{g} \to G$ *is a smooth map. Moreover if we identify* $T_0\mathfrak{g}$ *with* $\mathfrak{g}$*, then*

$$d\exp_0 = \mathrm{Id}.$$

Proof. Consider the map

$$\widetilde{\Phi} : \mathbb{R} \times G \times \mathfrak{g} \to G \times \mathfrak{g}, \quad (t, g, X) \mapsto (g \cdot \exp(tX), X).$$

One can check that this is the flow on $G \times \mathfrak{g}$ corresponding to the left invariant vector field $(X, 0)$ on $G \times \mathfrak{g}$, thus it is smooth. It follows that exp is smooth, since it is the composition

$$\begin{aligned} &\mathfrak{g} \hookrightarrow \mathbb{R} \times G \times \mathfrak{g} \xrightarrow{\widetilde{\Phi}} G \times \mathfrak{g} \qquad \xrightarrow{\pi_1} G, \\ &X \longmapsto (1, e, X) \longmapsto (\exp(tX), X) \mapsto \exp(tX). \end{aligned}$$

Since $\exp(tX) = \phi_t^X(e) = \gamma_e^X(t)$, we get

$$\left.\frac{d}{dt}\right|_{t=0} \exp(tX) = X.$$

On the other hand,

$$\left.\frac{d}{dt}\right|_{t=0} \exp \circ tX = (d\exp)_0 \frac{d(Xt)}{dt} = (d\exp)_0 X.$$

We conclude that $(d\exp)_0$ equals to the identity map. □

Since $(d\exp)_0$ is bijective, we have

Corollary 2.4. exp *is a local diffeomorphism near* 0*, i.e. it is a diffeomorphism from a neighborhood of* $0 \in T_eG$ *to a neighborhood of* $e \in G$*.*

Remark. According to the examples above, we see that in general exp is not a global diffeomorphism. However, it is still reasonable to ask:

Is the exponential map $\exp : \mathfrak{g} \to G$ surjective?

Of course for exp to be surjective, a necessary condition is that G should be connected. It turns out that for any compact connected Lie group G, the exponential map is always surjective. However, the exponential map need not be surjective for non-compact connected Lie groups (e.g. $SL(2, \mathbb{R})$).

¶ The Baker-Campbell-Hausdorff formula.

As another consequence, we can prove

Proposition 2.5. *For any $X, Y \in \mathfrak{g}$, there is a smooth map $Z : (-\varepsilon, \varepsilon) \to \mathfrak{g}$ so that for all $t \in (-\varepsilon, \varepsilon)$,*

$$\exp(tX)\exp(tY) = \exp(t(X+Y) + t^2 Z(t)).$$

Proof. Since exp is a diffeomorphism near $0 \in \mathfrak{g}$, there is an $\varepsilon > 0$ so that the map

$$\varphi : (-\varepsilon, \varepsilon) \to \mathfrak{g}, \quad t \mapsto \varphi(t) = \exp^{-1}(\exp(tX)\exp(tY))$$

is smooth. Note that we can write φ as the composition

$$\mathbb{R} \xrightarrow{\gamma_e^X \times \gamma_e^Y} G \times G \xrightarrow{\mu} G \xrightarrow{\exp^{-1}} \mathbb{R}.$$

According to Lemma 1.3 in Lecture 17, $d\mu_{e,e}(X, Y) = X + Y$. It follows

$$\varphi'(0) = (d\exp_0)^{-1}(\dot{\gamma}_e^X(0) + \dot{\gamma}_e^Y(0)) = X + Y.$$

Since $\varphi(0) = 0$, by Taylor's theorem,

$$\varphi(t) = t(X+Y) + t^2 Z(t)$$

for some smooth function Z. (Why? Can you write an explicit expression of Z involving integrals and derivatives of φ to show the smoothness?) □

Remark. The mysterious function Z is explicitly given by the Baker-Campbell-Hausdorff formula:

$$Z(t) = \frac{1}{2}[X, Y] + \frac{t}{12}([X, [X, Y]] - [Y, [X, Y]]) + \frac{t^2}{24}[X, [Y, [X, Y]]] + \cdots$$

For a proof, c.f. my Lie group notes.

¶ The naturality of exp**.**

Next we relates the exponential maps with the Lie group/Lie algebra homomorphisms. It turns out that exp is natural in the following sense:

Proposition 2.6 (exp is Natural)**.** *Given any Lie group homomorphism $\varphi : G \to H$, the diagram*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\varphi} & \mathfrak{h} \\ \downarrow{\scriptstyle \exp_G} & & \downarrow{\scriptstyle \exp_H} \\ G & \xrightarrow{\varphi} & H \end{array}$$

commutes, i.e. $\varphi \circ \exp_G = \exp_H \circ (d\varphi)$.

Proof. Left as an exercise. □

As an application, we have

Corollary 2.7. *If G is connected, any Lie group homomorphism $\varphi : G \to H$ is determined by the induced Lie algebra homomorphism $d\varphi : \mathfrak{g} \to \mathfrak{h}$.*

Proof. Left as an exercise. □

¶ [Reading Material] **Define Lie bracket on T_eG directly.**

As we have seen, each element $g \in G$ gives rise to an automorphism

$$c(g) : G \to G, \quad x \mapsto gxg^{-1}.$$

Notice that $c(g)$ maps e to e, its differential at e gives us a linear map

$$\mathrm{Ad}_g = (dc(g))_e : T_eG \to T_eG.$$

In other words, we get a map (the *adjoint representation* of the Lie group G)

$$\mathrm{Ad} : G \to \mathrm{End}(T_eG), \quad g \mapsto \mathrm{Ad}_g.$$

Note that $\mathrm{Ad}(e)$ is the identity map in $\mathrm{End}(T_eG)$. Moreover, since $\mathrm{End}(T_eG)$ is a linear space, its tangent space at Id can be identified with $\mathrm{End}(T_eG)$ itself in a natural way. Taking derivative again at e, we get (the *adjoint representation* of the Lie algebra $\mathfrak{g}$)

$$\mathrm{ad} : T_eG \to \mathrm{End}(T_eG).$$

Applying the naturality of exp to the Lie group homomorphism $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ and to the conjugation map $c(g) : G \to G$, we have

Proposition 2.8. (1) $\mathrm{Ad}(\exp(tX)) = \exp(t\mathrm{ad}(X))$.
(2) $g(\exp tX)g^{-1} = \exp(t\mathrm{Ad}_g X)$.

Now we show that ad is nothing else but the Lie bracket operation:

Theorem 2.9. *We have* $\mathrm{ad}(X)(Y) = [X, Y]$.

Proof. Since Ad_g is the differential of $c(g)$, we have

$$\mathrm{Ad}(\exp tX)Y = \frac{d}{ds}\Big|_{s=0} c(\exp tX)\exp sY = \frac{d}{ds}\Big|_{s=0} \exp(tX)\exp(sY)\exp(-tX).$$

So for any $f \in C^\infty(G)$, according to Proposition 2.8,

$$\begin{aligned}(\mathrm{ad}(X)Y)f &= (\frac{d}{dt}\Big|_{t=0} (\mathrm{Ad}(\exp tX)Y))f \\ &= \frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} f(\exp(tX)\exp(sY)\exp(-tX)) \\ &= \frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} f(\exp(tX)\exp(sY)) + \frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} f(\exp(sY)\exp(-tX)) \\ &= XYf(e) - YXf(e) = [X, Y]_e(f).\end{aligned}$$

□