LECTURE 20: LIE GROUP ACTIONS

1. Smooth actions

¶ Smooth actions.

As we mentioned, Lie groups are "smooth families of symmetries". In this lecture, we will regard Lie groups as the symmetry groups acting on smooth manifolds, and discuss basic facts from this point of view.

Let M be a smooth manifold. Then the "largest symmetry group" of M is $\mathrm{Diff}(M)$, the group of diffeomorphisms on M, which is not a Lie group since it is "too large": its dimension is infinite. [1] Now let G be a Lie group. What do we mean when we say G is a symmetry group of M? G must be a subgroup of $\mathrm{Diff}(M)$! So the following definition is natural:

Definition 1.1. Let G be a Lie group and M a smooth manifold.

(1) An *action* of a G on M is a homomorphism of groups $\tau : G \to \mathrm{Diff}(M)$. In other words, for any $g \in G$, $\tau(g)$ is a diffeomorphism from M to M such that
$$\tau(g_1 g_2) = \tau(g_1) \circ \tau(g_2).$$
(2) The action τ of G on M is *smooth* if the evaluation map
$$\mathrm{ev} : G \times M \to M, \quad (g, m) \mapsto \tau(g)(m)$$
is smooth. For simplicity we will denote $\tau(g)(m)$ by $g \cdot m$.

(3) A smooth manifold M endowed with a smooth G-action is called a *G-manifold.*

Remark. What we defined above is the *left action.* One can also define a *right action* to be an *anti*-homomorphism $\hat{\tau} : G \to \mathrm{Diff}(M)$, i.e. such that
$$\hat{\tau}(g_1 g_2) = \hat{\tau}(g_2) \circ \hat{\tau}(g_1).$$
Any left action τ can be converted to a right action $\hat{\tau}$ by letting $\hat{\tau}(g)(m) = \tau(g^{-1})(m)$.

¶ Examples of smooth actions.

Example. $\mathrm{GL}(n, \mathbb{R})$ (and thus any linear group) acts on $\mathbb{R}^n$ as linear transformations:
$$\forall X \in \mathrm{GL}(n, \mathbb{R}) \quad \rightsquigarrow \quad X : \mathbb{R}^n \to \mathbb{R}^n, v \mapsto Xv.$$
Can you write down a right action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$? This is why subgroups of $\mathrm{GL}(n, \mathbb{R})$ are called linear Lie groups.

Note that this induces an action of $O(n)$ on S^{n-1}.

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[1] However, one can regard $\mathrm{Diff}(M)$ as an infinite dimensional Lie group. Can you guess what is the Lie algebra associated to $\mathrm{Diff}(M)$? It is $\Gamma^\infty(TM)$, the space of smooth vector fields on M!

Example. We can define an S^1-action on S^2 by letting

$$\forall \theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z} \quad \rightsquigarrow \quad r_\theta(x^1, x^2, x^3) = (x^1 \cos\theta - x^2 \sin\theta, x^1 \sin\theta + x^2 \cos\theta, x^3).$$

(This action can be viewed as the restriction of the linear action of $\mathrm{GL}(3, \mathbb{R})$ on $\mathbb{R}^3$ to a suitable subgroup (write down it!) that is diffeomorphic to S^1. Since the action preserves S^2, it gives an smooth action on S^2.) It is not hard to see that these r_θ's are diffeomorphisms on S^2. Geometrically they are "counterclockwise rotations" with respect to the x^3-axis.

Example. Any Lie group G acts on itself by many ways, e.g. by left multiplication, by right multiplication and by conjugation. For example, the conjugation action of G on G is given by

$$g \in G \quad \rightsquigarrow \quad c(g) : G \to G, x \mapsto gxg^{-1}.$$

More generally, any Lie subgroup H of G can act on G by left multiplication, by right multiplication and by conjugation.

Example. Any Lie group G acts on its Lie algebra $\mathfrak{g} = T_eG$ by the *adjoint action*:

$$g \in G \quad \rightsquigarrow \quad \mathrm{Ad}_g = (dc(g))_e : \mathfrak{g} \to \mathfrak{g}.$$

For example, one can show that the adjoint action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathfrak{gl}(n, \mathbb{R})$ is given by

$$X \in \mathrm{GL}(n, \mathbb{R}) \quad \rightsquigarrow \quad \mathrm{Ad}_X : \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R}), A \mapsto XAX^{-1}.$$

Example. Let X be a complete vector field on M. Then the flow

$$\rho : \mathbb{R} \to \mathrm{Diff}(M), \quad t \mapsto \rho_t = \phi_t^X$$

is a smooth action of $\mathbb{R}$ on M. (We have called the image of ρ a one-parameter subgroup of $\mathrm{Diff}(M)$.)

Conversely, given any every smooth action of $\mathbb{R}$ on M, we can define a smooth vector field X on M by letting $X(m) = \dot\gamma_m(0)$, where $\gamma_m(t) := \rho_t(m)$, so that the $\mathbb{R}$-action is exactly the flow of X.

¶ The induced vector field.

The last example above can be extended to more general smooth Lie group actions easily: for any $X \in \mathfrak{g}$, there is a one-parameter subgroup $\exp(tX)$ in G.

Definition 1.2. Suppose Lie group G acts smoothly on M. For any $X \in \mathfrak{g}$, the *induced vector field* X_M on M associated to X is

$$X_M(m) = \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot m.$$

Note that the integral curve of X_M starting at $m \in M$ is $\gamma_m(t) = \exp(tX) \cdot m$, since by definition, $\gamma_m(0) = m$ and

$$\dot\gamma_m(t) = \frac{d}{dt}(\exp tX \cdot m) = \left.\frac{d}{ds}\right|_{s=0} (\exp sX \circ \exp tX \cdot m) = X_M(\gamma_m(t)).$$

In other words, the flow generated by X_M is $\rho_t = \tau(\exp(tX))$.

Remark. So from any smooth Lie group action of G on M we get a map

$$d\tau : \mathfrak{g} \to \Gamma^\infty(M), \quad X \mapsto X_M.$$

This can be viewed as the differential of the map $\tau : G \to \mathrm{Diff}(M)$. One can prove that $d\tau$ is a Lie algebra anti-homomorphism. It is called the *infinitesimal action* of $\mathfrak{g}$ on M.

2. Orbits and the quotient space

¶ Orbits and stabilizer.

Definition 2.1. Let $\tau : G \to \mathrm{Diff}(M)$ be a smooth action.

(1) The *orbit* of G through $m \in M$ is

$$G \cdot m = \{g \cdot m \mid g \in G\} \subset M.$$

(2) The *stabilizer* (also called the *isotropic subgroup*) of $m \in M$ is the subgroup

$$G_m = \{g \in G \mid g \cdot m = m\} < G.$$

Proposition 2.2. *Let $\tau : G \to \mathrm{Diff}(M)$ be a smooth action, $m \in M$. Then*

(1) *The orbit $G \cdot m$ is an immersed submanifold whose tangent space at m is*

$$T_m(G \cdot m) = \{X_M(m) \mid X \in \mathfrak{g}\}.$$

(2) *The stabilizer G_m is a closed Lie subgroup of G whose Lie algebra is*

$$\mathfrak{g}_m = \{X \in \mathfrak{g} \mid X_M(m) = 0\}.$$

Proof. (1) Let $\mathrm{ev}_m : G \to M$ be the map $\mathrm{ev}_m(g) = g \cdot m$. Then

$$\mathrm{ev}_m \circ L_g = \tau_g \circ \mathrm{ev}_m.$$

Taking derivative at $h \in G$, we get

$$(d\mathrm{ev}_m)_{gh} \circ (dL_g)_h = (d\tau_g)_{h \cdot m} \circ (d\mathrm{ev}_m)_h.$$

Since $(dL_g)_h$ and $(d\tau_g)_{h\cdot m}$ are bijective, the rank of $(d\mathrm{ev}_m)_{gh}$ equals the rank of $(d\mathrm{ev}_m)_h$ for any g and h. It follows that the map ev_m is of constant rank. By the constant rank theorem in Lecture 6, its image, $\mathrm{ev}_m(G) = G \cdot m$, is an immersed submanifold of M.

The tangent space of $G \cdot m$ at m is the image under $d\mathrm{ev}_m$ of $T_eG = \mathfrak{g}$. But for any $X \in \mathfrak{g}$ and any $f \in C^\infty(M)$, we have

$$(d\mathrm{ev}_m)_e(X)f = X_e(f \circ \mathrm{ev}_m) = \frac{d}{dt}\Big|_{t=0} f(\exp(tX) \cdot m) = X_M(m)(f).$$

So the conclusion follows.

(2) G_m is closed in G since $G_m = ev_m^{-1}(m)$. It is a subgroup of G since τ is a group homomorphism. By Cartan's closed subgroup theorem, G_m is a closed Lie subgroup of G. The Lie algebra of the Lie subgroup G_m is

$$\mathfrak{g}_m = \{X \in \mathfrak{g} \mid \exp(tX) \in G_m, \forall t \in \mathbb{R}\}.$$

It follows that $\exp(tX)\cdot m = m$ for $X \in \mathfrak{g}_m$. Taking derivative at $t=0$, we get

$$\mathfrak{g}_m \subset \{X \in \mathfrak{g} \mid X_M(m) = 0\}.$$

Conversely, if $X_M(m) = 0$, then $\gamma(t) \equiv m$, $t \in \mathbb{R}$, is an integral curve of the vector field X_M passing m. It follows that $\exp(tX)\cdot m = \gamma(t) = m$, i.e. $\exp(tX) \in G_m$ for all $t \in \mathbb{R}$. So $X \in \mathfrak{g}_m$. $\square$

Definition 2.3. We say a Lie group action of G on M is

1. *transitive* if there is only one orbit, i.e. $M = G \cdot m$.
2. *free* if $G_m = \{e\}$ for each $m \in M$.

For example, for any subgroup $H \subset G$, the natural action of H on G by left/right multiplications is free. We will see many examples of transitive actions below.

¶ The orbit space.

Suppose G acts on M smoothly. We will denote the set of orbits by M/G. For example, if the action is transitive, then M/G contains only one element. We will equip with M/G the quotient topology. This topology might be very bad in general, e.g. non-Hausdorff.

Example. Consider the natural action of $\mathbb{R}_{>0}$ on $\mathbb{R}$ by multiplications, then there are three orbits, $\{+, 0, -\}$. The open sets with respect to the quotient topology are

$$\mathcal{T} = \{\{+\}, \{-\}, \{+.-\}, \{+, 0, -\}, \emptyset\}.$$

So the quotient is not Hausdorff.

One of the major theorems in the theory of group actions is

Theorem 2.4. *Suppose a compact Lie group G acts on M smoothly, then*

1. (a) *Each orbit $G \cdot m$ is an embedded closed submanifold of M with*

   $$T_m(G\cdot m) = \{X_M(m) \mid X \in \mathfrak{g}\}.$$

   (b) *The orbit space M/G is Hausdorff.*
2. *If the action is free, then*
   (a) *The orbit space M/G is a smooth manifold.*
   (b) *The quotient map $\pi : M \to M/G$ is a submersion.*
3. *If the action is transitive, then for each $m \in M$, the map* [2]

   $$F : G/G_m \to M, \quad gG_m \mapsto g\cdot m$$

   is a diffeomorphism.

Remark. If G is not compact, the theorem still holds if we assume that the G-action is *proper*. For more details of the definition of proper action as well as the proof of the theorem, c.f. my Lie group course notes.

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[2] Note: G_m acts on G freely. Here we use the right action.

¶ Homogeneous spaces.

In particular, we see that if the G-action on M is transitive, then for any $m \in M$,

$$M \simeq G/G_m.$$

Definition 2.5. A smooth manifold on which a Lie group acts transitively is called a *homogeneous space.*

So by definition, homogeneous spaces are particularly nice smooth manifolds: they have lots of nice symmetries, so that "geometrically (in contrast to the word "topologically", which is true for all topological manifolds) all points are the same" in homogeneous spaces. Note that any homogeneous space is of the form G/H, where G is a Lie group, and H is a closed Lie subgroup of G. Note that for a given homogeneous space M, the Lie group G is not unique.

Example. The Euclidean group $E(n) = \mathbb{R}^n \ltimes O(n)$ is the set $\mathbb{R}^n \times O(n)$ equipped with the multiplication

$$(b, A) \cdot (b', A') = (b + Ab', AA').$$

$E(n)$ acts on $\mathbb{R}^n$ via

$$(b, A) \cdot x = b + Ax.$$

The action is obviously transitive. (This is known as *rigid body motions* in mechanics.) So $\mathbb{R}^n$ is a homogeneous space. Moreover, the stabilizer of the origin is $O(n)$. So

$$\mathbb{R}^n \simeq E(n)/\mathrm{O}(n).$$

Example. According to Gram-Schmidt, the natural action of $\mathrm{O}(n)$ on S^{n-1} is transitive. It follows that S^{n-1} is a homogeneous space. Moreover, if we choose m to be the "north pole" of S^{n-1}, then one can check that the stabilizer G_m is $\mathrm{O}(n-1)$. It follows

$$S^{n-1} \simeq \mathrm{O}(n)/\mathrm{O}(n-1).$$

Example. The special linear group $\mathbb{SL}(2, \mathbb{R})$ acts on the upper half plane $\mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$ by *Möbius transformations*,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

The action is transitive, and the stabilizer of $i \in \mathbb{H}$ is $\mathrm{SO}(2)$ (check). So

$$\mathbb{H} \simeq \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2).$$

Example. Let $k < n$. Consider

$$\mathrm{Gr}_k(n) = \{k \text{ dimensional linear subspaces of } \mathbb{R}^n\}.$$

Then $\mathrm{O}(n)$ acts transitively on $\mathrm{Gr}_k(n)$, and the stabilizer of $\mathbb{R}^k \times 0 \subset \mathbb{R}^n$ is $\mathrm{O}(k) \times \mathrm{O}(n-k)$. It follows

$$\mathrm{Gr}_k(n) \simeq \mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k))$$

The manifold $\mathrm{Gr}_k(n)$ is called a *Grassmannian manifold.* Note that $\mathrm{Gr}_1(n) = \mathbb{RP}^{n-1}$ is the real projective space.