LECTURE 21: TENSORS AND DIFFERENTIAL FORMS

1. Tensors as multi-linear maps

¶ Multi-linear maps.

Let $V_1, \cdots, V_k$ be finite dimensional vector spaces.

Definition 1.1. A function $T : V_1 \times \cdots \times V_k \to \mathbb{R}$ is called *multi-linear* if it is linear in each entry, i.e. for each i and any fixed vectors $v_1 \in V_1, \cdots, v_{i-1} \in V_{i-1}, v_{i+1} \in V_{i+1}, \cdots, v_k \in V_k$, the map

$$T_i : V_i \to \mathbb{R}, \quad v_i \mapsto T(v_1, \cdots, v_i, \cdots, v_k)$$

is linear.

Note that if T_1, T_2 are two multi-linear maps on $V_1 \times \cdots \times V_k$, so is their linear combinations. Thus the set of all multi-linear maps on $V_1 \times \cdots \times V_k$ is a vector space.

Example. For any $f^1 \in V_1^*, \cdots, f^k \in V_k^*$ (the dual spaces), we define

$$f^1 \otimes \cdots \otimes f^k : V_1 \times \cdots \times V_k \to \mathbb{R}, \qquad (v_1, \cdots, v_k) \mapsto f^1(v_1) \cdots f^k(v_k).$$

Obviously $f^1 \otimes \cdots \otimes f^k$ is a multi-linear map. Note that by definition, for each $1 \le i \le k$ and $\lambda \in \mathbb{R}$, we have

$$f^1 \otimes \cdots \otimes f^{i-1} \otimes \lambda f^i \otimes f^{i+1} \otimes \cdots \otimes f^k = \lambda f^1 \otimes \cdots \otimes f^{i-1} \otimes f^i \otimes f^{i+1} \otimes \cdots \otimes f^k.$$

Not surprisingly, any multi-linear map is a linear combination of these special maps:

Theorem 1.2. *Let $\{f_i^1, \cdots, f_i^{n(i)}\}$ be a basis of V_i^*. Then the set of multi-linear maps*

$$\{f_1^{i_1} \otimes f_2^{i_2} \otimes \cdots \otimes f_k^{i_k} \mid 1 \le i_j \le n(j)\}$$

form a basis of the vector space of multi-linear maps on $V_1 \times \cdots \times V_k$. In particular, $\dim \otimes^k V^* = n^k$.

Proof. We will denote by $\{e_1^i, \cdots, e_{n(i)}^i\}$ the basis in V that is dual to the basis $\{f_i^1, \cdots, f_i^{n(i)}\}$ of V_i^*. For any multi-index $I = (i_1, \cdots, i_k)$, we will denote $F^I = f_1^{i_1} \otimes f_2^{i_2} \otimes \cdots \otimes f_k^{i_k}$. Then the fact

$$F^I(e_{j_1}^1, \cdots, e_{j_k}^k) = \delta_{j_1, \cdots, j_k}^{i_1, \cdots, i_k}$$

implies that the multi-linear maps F^I's are linearly independent.

Moreover, for any multi-linear map T on $V_1 \times \cdots \times V_k$, if we let $T_I = T(e_{i_1}^1, \cdots, e_{i_k}^k)$, and consider the multi-linear map

$$S = T - \sum\nolimits_I T_I F^I,$$

then $S(e_{j_1}^1, \cdots, e_{j_k}^k) = 0$ for any multi-index $J = (j_1, \cdots, j_k)$. It follows from multi-linearity that $S \equiv 0$. In other words, $T = \sum T_I F^I$ is a linear combination of these F^I's. $\square$

Notation: We denote the vector space of multi-linear maps on $V_1 \times \cdots \times V_k$ by $V_1^* \otimes \cdots \otimes V_k^*$. Any element in this space is called a *k-tensor*.

Note that if $T \in V_1^* \otimes \cdots \otimes V_k^*$ and $S \in V_{k+1}^* \otimes \cdots \otimes V_{k+l}^*$, then we can define the *tensor product* $T \otimes S \in V_1^* \otimes \cdots \otimes V_{k+l}^*$ to be the tensor

$$(T \otimes S)(v_1, \cdots, v_{k+l}) = T(v_1, \cdots, v_k)S(v_{k+1}, \cdots, v_{k+l}).$$

By definition it is easy to see that $\otimes$ is a bilinear map and is associative:

$$(T \otimes S) \otimes R = T \otimes (S \otimes R).$$

In fact, we may simply regard elements in V_i^* as 1-tensors on V_i. Then the k-tensor $f^1 \otimes \cdots \otimes f^k$ that we used above is the tensor product of the 1-tensors $f^1, \cdots, f^k$. [One may need to check the consistency of the definition. What do I mean by consistency?]

Similarly we can define $V_1 \otimes \cdots \otimes V_k$ as the vector space of multi-linear maps on $V_1^* \times \cdots \times V_k^*$. By definition it has a basis $\{e^1_{i_1} \otimes e^2_{i_2} \otimes \cdots \otimes e^k_{i_k} \mid 1 \le i_j \le n(j)\}$.

Remark. For abstract vector spaces (which could be infinite dimensional), one can't regard V as $(V^*)^*$ but one can still define the tensor products algebraically. More precisely, we can define $V \otimes W$ to be the quotient space $V \otimes W = F(V \times W)/\sim$, where $F(V \times W)$ is the (infinite dimensional) *free vector space* over $V \times W$, and $\sim$ is the equivalence relations generated by $(v, w) \sim (v, w)$, $(c_1v_1 + c_2v_2, w) \sim c_1(v_1, w) + c_2(v_2, w)$ and $(v, c_1w_1 + c_2w_2) \sim c_1(v, w_1) + c_2(v, w_2)$. For any finite dimensional vector spaces V and W, one has a natural linear isomorphism $V \otimes W^* \simeq L(W, V)$, where $L(W, V)$ is the set of all linear maps from W to V. (Details will be left as an exercise.)

¶ Tensor powers of a vector space.

Now let V be an n-dimensional vector space, and V^* its dual space. We will call

$$\otimes^{l,k} V := (\otimes^l V) \otimes (\otimes^k V^*)$$

the space of (l, k)-*tensors* on V. In other words, $T \in \otimes^{l,k} V$ if and only if

$$T = T(\beta^1, \cdots, \beta^l, v_1, \cdots, v_k)$$

is multi-linear with respect to each $\beta^i \in V^*$ and each $v_j \in V$.

Remark. Note that $\otimes^{1,0} V = V$ and $\otimes^{0,1} V = V^*$. We will also abbreviate $\otimes^{k,0} V = \otimes^k V$. For the case $k = 0$, we denote $\otimes^0 V = \mathbb{R}$.

Next we will define a very useful operation on (l, k)-tensors.

Definition 1.3. For any $1 \le r \le l$ and $1 \le s \le k$, we define the (r, s)-*contraction* to be the map $C^r_s : \otimes^{l,k} V \to \otimes^{l-1,k-1} V$ given by

$$C^r_s(T)(\beta^1, \cdots, \beta^{k-1}, v_1, \cdots, v_{l-1}) = \sum_i T(\beta^1, \cdots, \beta^{r-1}, f^i, \beta^r, \cdots, \beta^{k-1}, v_1, \cdots, v_{s-1}, e_i, v_s, \cdots, v_{l-1})$$

where $\{e_1, \cdots, e_n\}$ is a basis of V, and $\{f^1, \cdots, f^n\}$ the dual basis.

One should check that this definition is independent of the choices of the basis $\{e_i\}$ of V. Moreover, $C^r_s(T)$ is the $(l-1, k-1)$-tensor obtained from T by pairing the r^{th} vector in T with the s^{th} co-vector in T:

Lemma 1.4. *Let T be an (l,k)-tensor. For $1 \le r \le l, 1 \le s \le k$, we have*

1. *The definition of C^r_s is independent of the choices of the basis $\{e_i\}$ of V.*
2. *For any $v_1, \cdots, v_l \in V$ and $\beta^1, \cdots, \beta^k \in V^*$,*

$$C^r_s(v_1 \otimes \cdots \otimes v_l \otimes \beta^1 \otimes \cdots \otimes \beta^k) = \beta^s(v_r) v_1 \otimes \cdots \otimes \widehat{v_r} \otimes \cdots \otimes v_l \otimes \beta^1 \otimes \cdots \otimes \widehat{\beta^s} \otimes \cdots \otimes \beta^k,$$

where $\widehat{\cdot}$ means "remove the corresponding entry".

Proof. Left as an exercise. □

Example. For example, if $v, w \in V$ and $\alpha, \beta, \gamma \in V^*$, one has

$$C^1_2(v \otimes w \otimes \alpha \otimes \beta \otimes \gamma) = \beta(v) w \otimes \alpha \otimes \gamma.$$

To see this, we compute by definition:

$$\begin{aligned} C^1_2(v \otimes w \otimes \alpha \otimes \beta \otimes \gamma)(\beta^1, v_1, v_2) &= \sum_i v \otimes w \otimes \alpha \otimes \beta \otimes \gamma(f^i, \beta^1, v_1, e_i, v_2) \\ &= \sum_i f^i(v)\beta^1(w)\alpha(v_1)\beta(e_i)\gamma(v_2) \\ &= \left(\sum_i f^i(v)\beta(e_i)\right)\beta^1(w)\alpha(v_1)\gamma(v_2) \\ &= \beta(v)\beta^1(w)\alpha(v_1)\gamma(v_2) \\ &= \beta(v) w \otimes \alpha \otimes \gamma(\beta^1, v_1, v_2). \end{aligned}$$

2. Linear p-forms

¶ Symmetric and anti-symmetric tensors.

Now let's fix a vector space V, and consider a k-tensor T on V, i.e. $T \in \otimes^k V^* = \otimes^{0,k} V$.

Definition 2.1. Let $T \in \otimes^k V^*$ be a k-tensor on V.

1. We say T is *symmetric* if for any permutation σ of $(1, 2, \cdots, k)$,
$$T(v_1, \cdots, v_k) = T(v_{\sigma(1)}, \cdots, v_{\sigma(k)}).$$
2. We say T is *alternating* (or a linear *k-form*) if it is skew-symmetric, i.e.
$$T(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = -T(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k).$$
for all $v_1, \cdots, v_k \in V$ and any $1 \le i \ne j \le k$

Example.
- An inner product on V is a *positive* symmetric 2-tensor.
- det is a linear n-form on $\mathbb{R}^n$.

We will denote the vector space of k-forms by $\Lambda^k V^*$. Note that $\Lambda^k V^*$ is not a brand new space: it is a linear subspace of $\otimes^k V^*$. We will set $\Lambda^1 V^* = \otimes^1 V^* = V^*$ and $\Lambda^0 V^* = \mathbb{R}$.

Recall that a permutation $\sigma \in S_k$ is called *even* or *odd*, depending on whether it is expressible as a product of an even or odd number of simple transpositions. For any k-tensor T and any $\sigma \in S_k$, we define another k-tensor T^σ by

$$T^\sigma(v_1, \cdots, v_k) = T(v_{\sigma(1)}, \cdots, v_{\sigma(k)}).$$

Clearly

- For all k-tensor T, $(T^\sigma)^\pi = T^{\pi \circ \sigma}$ for all $\sigma, \pi \in S_k$.
- A k-tensor T is symmetric if and only if $T^\sigma = T$ for all $\sigma \in S_k$.
- A k-tensor T is a k-form if and only if $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$, where $(-1)^\sigma = 1$ if σ is even, and $(-1)^\sigma = -1$ if σ is odd.

¶ Anti-symmetrization.

For any k-tensor T on V, we consider the *anti-symmetrization map*

$$\mathrm{Alt}(T) = \frac{1}{k!} \sum_{\pi \in S_k} (-1)^\pi T^\pi.$$

Lemma 2.2. *The map* Alt *is a projection from $\otimes^k V^*$ to $\Lambda^k V^*$, i.e. it is a linear map satisfying*

(1) *For any $T \in \otimes^k V^*$, $\mathrm{Alt}(T) \in \Lambda^k V^*$.*
(2) *For any $T \in \Lambda^k V^*$, $\mathrm{Alt}(T) = T$.*

Proof. (1) For any $T \in \otimes^k V^*$ and any $\sigma \in S_k$,

$$[\mathrm{Alt}(T)]^\sigma = \frac{1}{k!} \sum_{\pi \in S_k} (-1)^\pi (T^\pi)^\sigma = \frac{1}{k!} (-1)^\sigma \sum_{\pi \in S_k} (-1)^{\sigma \circ \pi} T^{\sigma \circ \pi} = (-1)^\sigma \mathrm{Alt}(T).$$

(2) If $T \in \Lambda^k V^*$, then each summand $(-1)^\pi T^\pi$ equals T. So $\mathrm{Alt}(T) = T$ since $|S_k| = k!$. □

We will need

Lemma 2.3. *Let T, S, R be k-, l-, and m-forms respectively. Then*

(1) $\mathrm{Alt}(T \otimes S) = (-1)^{kl} \mathrm{Alt}(S \otimes T)$.
(2) $\mathrm{Alt}(\mathrm{Alt}(T \otimes S) \otimes R) = \mathrm{Alt}(T \otimes S \otimes R) = \mathrm{Alt}(T \otimes \mathrm{Alt}(S \otimes R))$.

Proof. Exercise. □

¶ The wedge product.

Now we can define a "product operation" for linear forms:

Definition 2.4. The *wedge product* of $T \in \Lambda^k V^*$ and $S \in \Lambda^l V^*$ is the $(k+l)$-form

$$T \wedge S = \frac{(k+l)!}{k! l!} \mathrm{Alt}(T \otimes S).$$

The wedge product operation satisfies

Proposition 2.5. *The wedge product operation* $\wedge : (\Lambda^k V^*) \times (\Lambda^l V^*) \to \Lambda^{k+l} V^*$ *is*

(1) *Bi-linear:* $(T, S) \mapsto T \wedge S$ *is linear in* T *and in* S.
(2) *Anti-commutative:* $T \wedge S = (-1)^{kl} S \wedge T$.
(3) *Associative:* $(T \wedge S) \wedge R = T \wedge (S \wedge R)$.

Proof. (1) follows from Definition 2.4. (2) follows from Lemma 2.3(1). (3) follows from Definition 2.4 and Lemma 2.3(2). □

So it makes sense to talk about wedge products of three or more linear forms. For example, if $T \in \Lambda^k V^*, S \in \Lambda^l V^*$ and $R \in \Lambda^m V^*$, then we have

$$T \wedge S \wedge R = \frac{(k+l+m)!}{k!l!m!} \mathrm{Alt}(T \otimes S \otimes R).$$

One can easily extend this to wedge products of more than three linear forms. In particular, by definition we have: if $f^1, \cdots, f^k \in V^*$, then

$$f^1 \wedge \cdots \wedge f^k = k! \mathrm{Alt}(f^1 \otimes \cdots \otimes f^k).$$

As a consequence,

Proposition 2.6. *For any* $f^1, \cdots, f^k \in V^*$ *and* $v_1, \cdots, v_k \in V$,

$$(f^1 \wedge \cdots \wedge f^k)(v_1, \cdots, v_k) = \det(f^i(v_j)).$$

Proof. We have

$$\begin{aligned}
(f^1 \wedge \cdots \wedge f^k)(v_1, \cdots, v_k) &= k!\ \mathrm{Alt}(f^1 \otimes \cdots \otimes f^k)(v_1, \cdots, v_k) \\
&= \sum_{\sigma \in S_k} (-1)^\sigma f^1(v_{\sigma(1)}) \cdots f^k(v_{\sigma(k)}) \\
&= \det((f^i(v_j))).
\end{aligned}$$

□

¶ The vector space of linear k-forms.

Now we are ready to prove

Theorem 2.7. *Let* $\{f^1, \cdots, f^n\}$ *be a basis of* V^*. *Then the set of* k*-forms*

$$\{f^{i_1} \wedge f^{i_2} \wedge \cdots \wedge f^{i_k} \mid 1 \le i_1 < i_2 < \cdots < i_k \le n\}$$

form a basis of $\Lambda^k V^*$. *In particular,* $\dim \Lambda^k V^* = \binom{n}{k}$.

Proof. Again we denote by $\{e_1, \cdots, e_n\}$ the dual basis in V. For any multi-index $I = (i_1, \cdots, i_k)$ with $i_1 < \cdots < i_k$, we let $\Omega^I = f^{i_1} \wedge \cdots \wedge f^{i_k}$. Then for any multi-index $J = (j_1, \cdots, j_k)$ with $j_1 < \cdots < j_k$,

$$\Omega^I(e_{j_1}, \cdots, e_{j_k}) = \det((f^{i_r}(e_{j_s}))_{1 \le r,s \le k}) = \delta^{i_1, \cdots, i_k}_{j_1, \cdots, j_k}.$$

It follows that these Ω^I's are linearly independent.

Moreover, since any $T \in \Lambda^k V^*$ is a k-tensor, we can write $T = \sum_I T_I F^I$, where $I = (i_1, \cdots, i_k)$ runs over all $1 \le i_1, \cdots, i_k \le n$, and F^I is as in the proof of Theorem 1.2. Note that $\Omega^I =$

$k!\text{Alt}(F^I)$. Here, the indices I need not be increasing, but we have: $\Omega^I = 0$ if two indices in I equal, and $\Omega^I = \pm\Omega^{I'}$ if I contains no equal indices, where I' is the re-arrangement of I in increasing order. So

$$T = \text{Alt}(T) = \sum_{\text{all } I} T_I \text{Alt}(F^I) = \frac{1}{k!} \sum_{I \text{ increasing}} \left(\sum_{I'=I \text{ as sets}} (\pm T_{I'}) \right) \Omega^I$$

is a linear combination of Ω^I with I's being only increasing indices. □

Remark. As an immediate consequence, we see

- $\dim \Lambda^n(V^*) = 1$.
  - So any n-form on an n-dimensional vector space V is a multiple of the non-trivial n-form "det".
- Moreover, for $k > n$, $\Lambda^k(V^*) = 0$.

¶ The interior product and the pull-back.

Finally we define two more operators on linear k-forms.

Definition 2.8. The *interior product* of a vector $v \in V$ with a linear k-form $\alpha \in \Lambda^k(V^*)$ is the $(k-1)$-covector

$$\iota_v\alpha(X_1, \cdots, X_{k-1}) := \alpha(v, X_1, \cdots, X_{k-1}).$$

Definition 2.9. Let $L : W \to V$ be linear. The *pullback* $L^* : \Lambda^k(V^*) \to \Lambda^k(W^*)$ is defined to be

$$(L^*\alpha)(X_1, \cdots, X_k) := \alpha(L(X_1), \cdots, L(X_k))$$

The following proposition will be important in the rest of this semester. The proof is left as an exercise.

Proposition 2.10. *Let α be a linear k-form on V and β a linear l-form on V. Then*

(1) *For any $v \in V$, $\iota_v\iota_v\alpha = 0$.*
(2) *For any $v \in V$, $\iota_v(\alpha \wedge \beta) = (\iota_v\alpha) \wedge \beta + (-1)^k\alpha \wedge \iota_v\beta$.*
(3) *For any linear map $L : W \to V$, $L^*(\alpha \wedge \beta) = L^*\alpha \wedge L^*\beta$.*

Proof. Left as an exercise. □

3. Reading: Tensors fields and differential forms on smooth manifolds

¶ Cotangent spaces.

Let M be a smooth manifold. We have associated to each $p \in M$ a vector space T_pM. If we take any local chart (φ, U, V) around p, then we can write down an explicit basis for T_pM:

$$\partial_i|_p : C^\infty(U) \to \mathbb{R}, \quad \partial_i|_p(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)), \qquad (1 \leq i \leq n).$$

Note that not only the $\partial_i|_p$'s form a basis for the tangent space T_pM, but in fact ∂_i's are smooth vector fields on U, and for any $q \in U$, the $\partial_i|_q$'s form a basis for the tangent space T_qM.

Now let's study the dual space T_p^*M of T_pM. We introduced T_p^*M in PSet 3-1-6. It is called the *cotangent space* of M at p, and elements in T_p^*M are called *cotangent vectors* at p. It is also quit easy to write down an explicit basis of T_p^*M, (and in fact a basis of T_q^*M, for each $q \in U$, varying smoothly in q), in any given local chart (φ, U, V): We first note that for each $1 \le i \le n$,

$$x^i \circ \varphi : U \to \mathbb{R}$$

is a smooth function on U. The differential of this function, which we will denote by dx^i for simplicity, is a linear map (when restricted to any $q \in U$)

$$dx^i|_q : T_qM = T_qU \to T_{x^i\circ\varphi(q)}\mathbb{R} = \mathbb{R}.$$

In other words, each $dx^i|_q$ is an element in T_q^*M. More over, by definition,

$$dx^i|_q(\partial_j|_q) = \partial_j|_q(x^i \circ \varphi) = \delta^i_j.$$

So we conclude

Proposition 3.1. *In any local chart (φ, U, V), $\{dx^i|_q : 1 \le i \le n\}$ is a basis of T_q^*M. Moreover, this basis is the dual basis to the basis $\{\partial_i|_q : 1 \le i \le n\}$ of T_qM.*

In fact, for any $f \in C^\infty(U)$, by the same way we get a linear map $df_q : T_qM \to \mathbb{R}$. In other words, we get a cotangent vector $df_q \in T_q^*M$. By definition, $df_p(\partial_i|_p) = \partial_i|_p(f)$. It follows

$$df_p = (\partial_1|_pf)dx^1|_p + \cdots + (\partial_n|_pf)dx^n|_p,$$

and moreover, for any $X \in \Gamma^\infty(TU)$,

$$df(X) = Xf,$$

where both sides are regarded as functions on U. We will call df a 1-form on U.

¶ Tensor fields on smooth manifolds.

Now we are ready to define (Compare: the definition of vector fields on manifolds)

Definition 3.2. An (l,k)-*tensor field* T on M is an assignment that assigns to each point $p \in M$ an (l,k)-tensor $T_p \in \otimes^{l,k}T_pM$.

Remark. By definition, T is a tensor at if and only if it is point-wise linear in each entry. It follows that T is tensor field on M if and only if it is "function-linear" in each entry. So it is more than a multi-linear map, i.e., we also have (where ω's are 1-forms on U, and X's are vector fields on U)

$$T(f_1\omega^1, \cdots, f_l\omega^l, g^1X_1, , \cdots, g^kX_k) = f_1\cdots f_lg^1\cdots g^kT(\omega_1, \cdots, \omega_l, X_1, \cdots, X_k).$$

If we fix any local chart (φ, U, V) near p, then we can write

$$T_p = \sum T^{i_1\cdots i_l}_{j_1\cdots j_k}\ \partial_{i_1}|_p \otimes \cdots \otimes \partial_{i_l}|_p \otimes dx^{j_1}|_p \otimes \cdots \otimes dx^{j_k}|_p,$$

where $T^{i_1\cdots i_l}_{j_1\cdots j_k}$'s are constants (which depends on p). In other words, in any coordinate chart U one can write

$$T = \sum T^{i_1\cdots i_l}_{j_1\cdots j_k}\partial_{i_1} \otimes \cdots \otimes \partial_{i_l} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_k},$$

where $T^{i_1\cdots i_l}_{j_1\cdots j_k}$'s are functions on U.

Definition 3.3. We say a (l,k)-tensor field T on M is *smooth* if in any coordinate chart U, the functions $T^{i_1\cdots i_l}_{j_1\cdots j_k}$'s are smooth.

Note that when $(l,k)=(1,0)$, we will get a smooth vector field on M. The set of all smooth (l,k)-tensors is denoted by $\Gamma^\infty(\otimes^{l,k}TM)$. Again this is an infinite dimensional vector space.

Remark. The coefficient functions $T^{i_1\cdots i_l}_{j_1\cdots j_k}$'s are only defined in local charts. If one uses another chart U', one gets another set of coefficient functions (even if at the same point).

Example. A symmetric positive smooth $(0,2)$-tensor field g on M is called a *Riemannian metric* on M. Locally a Riemannian metric is of the form

$$g=\sum g_{ij}(x)dx^i\otimes dx^j,$$

where $(g_{ij}(x))$ is a positive definite symmetric matrix depending smoothly on x. We have seen the existence of Riemannian metric on any smooth manifold in PSet 3-2-5.

¶ Differential forms on smooth manifolds.

Similarly one can define smooth k-forms on a smooth manifold M:

Definition 3.4. A *k-form* ω on a smooth manifold M is an assignment that assigns to each point $p\in M$ a linear k-form $\omega_p\in\Lambda^kT_p^*M$. A k-form ω is *smooth* if locally one can write

$$\omega=\sum_I\omega_I dx^I=\sum_I\omega_{i_1,\cdots,i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k},$$

where the summation is over increasing k-tuples $I=\{1\le i_1<\cdots<i_k\le n\}$, and each $\omega_I\in C^\infty(U)$.

Since k-forms will be frequently used in the rest of this course, we will denote the set of all smooth k-forms by $\Omega^k(M)$ (instead of the lengthy expression $\Gamma^\infty(\Lambda^kT^*M)$). Note that any smooth function on M can be viewed as a smooth 0-form. So

$$\Omega^0(M)=C^\infty(M).$$

Since there is no linear k-form on T_pM for $k>n=\dim M$, we get

$$\Omega^k(M)=0,\quad \forall k>n.$$

Note that if $\omega\in\Omega^k(M)$, and $X_1,\cdots,X_k\in\Gamma^\infty(TM)$, then $\omega(X_1,\cdots,X_k)\in C^\infty(M)$.

¶ Operations on differential forms on smooth manifolds.

Of course the pointwise operations for linear k-forms that we learned last time still make sense for differential forms on manifolds. So on differential forms we have the following operations:

- The *wedge product* $\wedge:\Omega^k(U)\times\Omega^l(U)\to\Omega^{k+l}(U)$.
  - For example,

    $$(dx^1+2dx^2)\wedge(dx^1\wedge dx^2-dx^2\wedge dx^3+3dx^1\wedge dx^3)=-7dx^1\wedge dx^2\wedge dx^3.$$
- For any $X\in\Gamma^\infty(TU)$, one has the *interior product* $\iota_X:\Omega^k(U)\to\Omega^{k-1}(U)$.

  - For example,
$$\iota_X(dx^{i_1}\wedge\cdots\wedge dx^{i_k})=\sum_r(-1)^{r-1}dx^{i_r}(X)dx^{i_1}\wedge\cdots\wedge\widehat{dx^{i_r}}\wedge\cdots\wedge dx^{i_k}.$$

- For any smooth map $\varphi: U'\to U$, one has the *pull-back* $\varphi^*:\Omega^k(U)\to\Omega^k(U')$.
  - This is defined pointwise via the linear map $d\varphi_p: T_pU'\to T_{\varphi(p)}U$. So if $\omega\in\Omega^k(U)$, then
$$(\varphi^*\omega)_p(X_1,\cdots,X_k)=\omega_{\varphi(p)}(d\varphi_p(X_1),\cdots,d\varphi_p(X_k)).$$
    Note: if $k=0$, then φ^* is exactly the pull-back $\varphi^*: C^\infty(U)\to C^\infty(U')$ on functions.

These operations are all linear (where $\wedge$ is bilinear). Here we list some important properties of these operations.

Proposition 3.5. *Suppose* $\omega\in\Omega^k(U),\eta\in\Omega^l(U)$, $X\in\Gamma^\infty(TU)$ *and* $\varphi\in C^\infty(U',U)$ *and* $\psi\in C^\infty(U,\widetilde{U})$. *Then*

(1) $\omega\wedge\eta=(-1)^{kl}\eta\wedge\omega$.
(2) $\varphi^*(\omega\wedge\eta)=\varphi^*\omega\wedge\varphi^*\eta$.
(3) $\iota_X(\omega\wedge\eta)=(\iota_X\omega)\wedge\eta+(-1)^k\omega\wedge\iota_X\eta$.
(4) $\iota_X\circ\iota_X=0$.
(5) $(\psi\circ\varphi)^*=\psi^*\circ\varphi^*$.

Proof. All follows from definitions and corresponding results for linear differential forms. $\square$